CHAPTER 2
CRITICAL DEVELOPMENTS

The shortest answer to the question concerning the type of unification which is present in aggregates is the direct reference to the phenomena. And really we deal here with ultimate facts. But this does not dispense us from the task of considering this type of union more accurately, and of highlighting its characteristic differences from other types of union, especially since often enough there have been false characterizations and confusions with other types of relation.

We want to examine, to this end, a series of theories that are possible and in part really presented, each of which characterizes the collective union in a different way, and on that basis, tries to explain also in a different way the origin of the concepts of plurality and number.

THE COLLECTIVE UNION AND THE UNIFICATION OF THE PARTIAL PHENOMENA IN EACH MOMENT’S TOTAL CONSCIOUSNESS [?] 

The union of the ideas into an aggregate, one could say, hardly deserves the name of a union. For, what is there when we speak of an aggregate of arbitrary objects? Nothing beyond the fact that these objects are together in our consciousness. The unity of the ideas of the aggregate consists therefore only in the belonging to the consciousness that puts them together /18/. This is nevertheless a fact to which one must pay attention; for by reflection on that fact the concepts in whose analysis we are interested here arise.

This view is obviously wrong. Very complex phenomena constitute at each moment the content of our total consciousness; but there are special interests leading to pick some ideas from this totality and to unify them collectively. And this happens without the disappearance from the consciousness of all the other ideas. If that view was right, then there would be at each moment only one aggregate, consisting of the totality of the available partial contents of our total consciousness; whereas, all the time and arbitrarily, we form various aggregates, we extend an already formed by adding new contents, or we make it smaller by dropping some of the contents, without thereby removing the dropped
elements from consciousness; briefly, we are aware of a spontaneity which would be otherwise unthinkable [?]

In addition, that view, if taken in its general and not further specified version, contains an absurdity. In fact, do not continua, with their infinite set of points, belong to the contents of our consciousness? Who could have really represented them ever in the form of an aggregate?

It is important to emphasize that to an aggregate (of a genuine idea of plurality) can belong as elements only contents such that we are conscious of them as noticed by themselves ["of themselves"?]; but all the other contents, which are noticed only as being nearby, and which either cannot at all be noticed by themselves (as the points of the continua) or which are not noticed by themselves just for the time being, -none of these can yield the elements out of which an aggregate is formed.

All this will perhaps find easy agreement, and the representative /19/ of the just criticized view should restrict his claim so that by "comprehensive consciousness" which unifies the ideas into a plurality, one should understand a particular act of the consciousness rather than consciousness in the broadest sense as totality of our psychological phenomena. Thus, accordingly, the unity in question would be the unity in a selecting and bringing-together act of the mind, or the unity of the interest, or something similar. We want to return later to a more accurate evaluation of such a revised theory.

THE COLLECTIVE TOGETHERNESS AND THE TEMPORAL SIMULTANEITY

We move now to the consideration of a new theory which argues as follows:

If an aggregate of contents is present to us, what else should be noticed aside from this: each content is there simultaneously with all the others. The temporal coexistence of the contents is indispensable for the representation of their plurality. Now, each compound act of thought requires, to be sure, the coexistence of its parts; but while in other cases in addition to the simultaneity still other relations or combinations, which unify the parts, are present, it is precisely the distinctive peculiarity of the idea of aggregate, that it does not contain anything but the simultaneous contents. Hence plurality in abstracto does not mean anything else but: simultaneous being given of any contents whatsoever.
This view is subject, as one can easily see, to precisely the same objections as the previous one, and still others. It would be superfluous to repeat the former; about the latter it is sufficient to emphasize that to represent contents simultaneously does not yet mean to represent contents as simultaneous. For example, in order that the idea of a melody arises, the individual sounds that constitute it must be related to one another. But each relation requires the simultaneous presence, in one act of consciousness, of the related contents. Therefore, also the sounds of the melody must be represented simultaneously. But not at all as simultaneous; quite to the contrary, they appear to us as being in a certain temporal sequence.

Not otherwise in the case in which we represent a plurality of objects. That we must represent the objects simultaneously, is certain; but that we do not represent them as simultaneous, and that rather special reflections are needed in order to notice that simultaneity of the objects, is immediately proved by reference to the internal experience.

Thus, we see that the collective togetherness cannot be described as temporal simultaneity.

**COLLECTION AND SUCCESSION**

A third view is equally based on time as a non-eliminable psychological factor. In direct contrast with the preceding view, it argues as follows:

Because of the discursive nature of our mind, not many contents, different from one another, can be thought simultaneously. Our consciousness can at each moment be concerned only with one object. Every relational and higher activity of the mind is possible only because the objects to which it is referred, are given temporally one after the other. Thus every complicated thought-formation, every whole composed of whatever parts, has come to be out of simple factors successively; we have always to do with stepwise processes and operations which, developing in time, always extend further and [?]. In particular therefore each collection presupposes a collecting, each number a counting, and hereby a temporal ordering is given necessarily of the objects brought-together or of the counted units. But there is still more.

The temporal succession, and nothing else, is what characterizes the plurality as plurality.
To be sure, there is a succession wherever contents stand in any relation, simple or compound, and thus are united into a global representation; but we can always then at the same time speak of a plurality; and this cannot however happen with regard to any relation changing from one case to another but with regard to that unique relation that occurs everywhere, namely the temporal sequence.

In the case of the conceivably loosest union, where arbitrary contents, for the rest unrelated (or conceived in abstraction from possible relations) are thought merely together, i.e. as plurality, there the succession, with which the contents entered the consciousness, becomes the only relation which still links the contents.

Therefore, what characterizes the plurality vis à vis other, more internal forms of composition, is the fact that in it the mere succession relates the contents, whereas in the others there are still additional relations at work. Accordingly it follows: plurality in the abstract is nothing else but succession, succession of any arbitrary contents noticed by themselves. And the number concepts represent the determined forms of plurality or of succession in the abstract.

To avoid dividing the reader's attention by individual criticisms that are not fruitful, I have preferred, rather than to criticize one after the other the individual authors who have held such or similar theories, to present as clearly and consequently as at all possible, the doctrine itself which they more or less clearly have in mind, and to exert on it the criticism.

/22/ The view to be opposed here rests on substantial psychological and logical errors.

In the first place it resorts to the psychological fact of the narrowness of consciousness, but exaggerating and falsely interpreting it. It is true that the number of particular contents to which we can at each moment pay attention is highly limited, indeed it shrinks, in the case of maximal concentration of the interest, to just one. But it is untrue that in one and the same moment we can never deal with more than one content.

Indeed, the fact of the relational and linking thinking, as well as in general of all the more complicated mental and emotional activities to which that theory refers, shows evidently the total absurdity of its conception. If at each moment only one content is present to our consciousness, how should we be able to notice even the simplest relation? If we
represent one term of the relation, then the other is either not yet in our consciousness or no longer in it. Now, we cannot link a content of which we are not aware, which therefore does not exist at all for us, with the unique content that is present and really given to us.

Thus, in this interpretation of the narrowness of consciousness, the reference to the temporal succession of the ideas to be related, rather than justifying the possibility of the relational thought, would make evident on the contrary its impossibility.

But does not experience teach (so perhaps the opponent replies) that we in fact can have always only one idea present, and that it is perhaps very feasible to relate it to past ones? Hence, by being past, an idea does not cease at all to exist.

One sees easily that such an answer would rest on misinterpretations of experience. One should not confuse present ideas with ideas of the present and past ideas with ideas of the past. Not every present idea is, as we must emphasize again, an idea of the present. Precisely all the ideas that refer to the past constitute an exception; for they are in reality present ideas. If for instance we recall a song that we heard yesterday, this recollection is in fact a present idea; it is only related by us to the past.

If this is kept in mind, then naturally one will not find any problem with the fact that we can link ideas of present content and ideas of past content. When we do this, all of them are available simultaneously in our consciousness, they are all present ideas. On the contrary, we cannot combine relationally past ideas either among themselves or with present ideas; for qua past they cannot return and remain forever behind.

The supposed empirical fact considered by the opponent would therefore reduce to that, whenever we conceive a plurality of contents, there would always be only one present, while all the others would show greater or smaller temporal diversities. Naturally then each whole of ideas, consisting of parts distinguished (noticed by themselves) should have originated by successive acts of noticing and relating the individual partial contents, while the whole itself, as a ready and full-grown one, would contain all the parts at the same time, except that provided with unequal temporal determinations.

Now it is certainly sure that already in the case of a very moderate number of contents a global notice of them is possible only by /24/ considering and retaining them successively or in very small groups. But shouldn't we in the case of two or three very different
contents have the ability to conceive them and to hold them combined in one act, without being necessary a successive passage from one to the other? I would not dare to deny this so confidently.

Be as it may, we can acknowledge as a fact that for the genesis of the ideas of sets (perhaps with a few exceptions) and of all ideas of number the temporal succession constitutes an indispensable psychological condition. Thus one is entirely justified in referring, if not to all, to almost all sets and numbers as results of processes, and to the extent that our will is herein involved, as results of activities, of operations of collecting or counting.

But this is also all that we can grant. Only this one thing, and nothing more has been proved: the succession in time constitutes an unremovable pre-requisite for the formation [weitaus] of most number concepts /25/ and concrete pluralities -as well as of all complex concepts in general. They have a temporal becoming, and through this every part of the resulting whole obtains a different temporal determination in our representation. But is it thereby also established that the temporal order enters the content of those concepts or even is the special relation which characterizes pluralities as such vis a vis other types of composition?

Frequently enough one was satisfied with such defective arguments, without observing that time exactly in the same way constitutes the foundation for every higher thinking, and one for example could infer with precisely the same right that the relation between premiss and conclusion is identical with their temporal sequence. But the form we gave to the time -theory for our purposes is already free of this obvious incompatibility. Its claim is only that the case of the aggregate (of the concrete plurality) is distinguished from that of any other compound whole by there being in it a mere succession of the partial contents, whereas in other wholes there are, in addition, still some other.

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1 Perhaps some readers are surprised by my expressing myself here in such a doubting mode, although confirming examples even for significantly greater numbers are easily adduced. In fact for example we grasp in the game of [domino] groups of ten to twelve dots in one glance, indeed we even learn quite immediately their number. But it must be considered that in such cases one cannot speak either of a real collecting or of a real counting. The numeral is here directly associated with the characteristic sensible appearance, and it is recalled each time by it without any conceptual mediation. In so large sets, as everyone can verify, a direct and proper collecting and counting is an impossibility. In quite small groups of two or three objects the thing is doubtful because the successive apprehensions of the elements may have occurred so rapidly that they [the apprehensions] had to escape [evn?] to the sharpened attention. Hence the cautious mode of my text.
connections. Therefore it does not simply argue: because counting requires temporal succession of the ideas, number is the synthetizing form in the abstract of what is successive; but it intends to be able to also show that the temporal sequence constitutes the only thing that is common to all instances of plurality, and for this reason it must be the foundation for the abstraction of this concept.

But even thus conceived, we cannot approve the theory. If it was right, then any intelligible distinction between the concepts of plurality and of succession, which probably nobody will seriously identify, would be lacking. For what sense would otherwise make to speak of a plurality of simultaneous contents? The genesis of the concept /26/ of the temporal coexistence would be, from this point of view, an unintelligible puzzle.

And in the same way we can regard any conceivable attempt to clarify the concepts of plurality and number by referring to the temporal succession, as apriori hopeless. The following simple consideration makes this obvious.

If the temporal succession had to contribute somehow to the content of the mentioned concepts, then it would not suffice to ascertain its existence in all the concrete cases; it should also constitute in all cases the object of special attention. But this is surely not the case. We do not always pay attention to the temporal relations, and precisely for that reason we are able to distinguish between a plurality simpliciter and a plurality of contents that come one after the other (or that are simultaneous). All over again on the one hand this mistake is made, on the other hand [...]; to perceive contents temporally following one another does not yet mean to perceive contents as temporally following one another, a thesis that we had to occasionally emphasize also with regard to simultaneous contents (p. 20). But it is of special importance to observe -and this is generally overlooked- that even where we focus on a temporal sequence of contents, pluralities already determined are in no way selected. This is done only by certain synthetizing psychic acts. To overlook them means precisely to leave out of consideration what is the true and only source of the concept of plurality, as well of the concept of number. Examples may clarify all this.

The clock makes a uniform "tic-tac"; I hear the individual strokes but it is not necessary that I attend to their temporal sequence. And even when I attend to it, this does not yet highlight any number of strokes, the latter are /27/ not unified by a synthetizing mind into
an aggregate. Or another example: the eyes wander around, in different directions, now focussing on this object, now on that object, awakening thus a variety of ideas.

But a special interest is necessary, if the temporal sequence is to be noticed by itself. And in order to retain by themselves, to order one after the other and to synthetize into an aggregate all or some of the objects noticed, to this special interests and special acts of noticing belong each time, directed towards those selected contents ant to no others. Hence even if the temporal sequence in which the objects are collected was continuously noticed, it would still remain always incapable of providing just by itself a foundation for the unity of the collective whole.

But since we cannot even grant that the temporal following one after the other enters at least as constant and all the time noticed part in the representation of every aggregate, it is clear that much less can it enter in the corresponding general concept (plurality, number).

Herbart is completely right in saying: "Number has nothing more in common with time than hundred other types of ideas, which also could be generated only in a gradual way[?]"; and Beneke expresses the same idea equally drastically and appropriately: "That time flows over counting cannot be rejected[?] by any proof: for over what would not flow time[?]"

If the question was just to describe the phenomenon that is before us when we conceive a

\[\text{Psychologie als Wissenschaft, Königsber 1824, II, 162}\]

\[\text{System der Logik, Berlin 1842, I, 279.}\]
plurality, then we certainly ought to mention the temporal modifications occurring in the individual contents, although they are not in general particularly noticed. But aside from the fact that exactly the same holds for every compound whole, in general a distinction must be made between the phenomenon as such and that for which it is useful to us or what it means for us; and accordingly also between the psychological description of a phenomenon and the account of its significance [Bedeutung]. The phenomenon is the foundation for the significance, but it is not the significance itself. If there is an aggregate of objects A, B, C, D in our mind, then, with regard to the successive process by which the whole emerges, perhaps finally only D will be given as sensation, while the other contents will appear merely as phantasies in a way that is modified temporally or also with regard to content.

If inversely we start from D towards A, the phenomenon is a different one. The logical significance eliminates all these distinctions. /29/ The modified contents serve as signs, as representatives for the unmodified ones. In forming the notion of aggregate, we do not pay attention to the changes that occur in the contents during the process of collecting; we intend to really keep them fixed and to unite them, and thus the logical content of that representation is not for instance D, the just seen C, the earlier seen B, till the most modified A, but nothing else as (A, B, C, D); the representation comprehends each one of the contents regardless of the temporal distinctions and the temporal order based on those distinctions.

Thus we see that time plays for our concepts only the role of a psychological precondition, and this in a twofold manner:

1) It is unavoidable that the partial notions united in the representation of the plurality or of the number be available at the same time in our consciousness.

2) Almost all the representations of plurality and at any rate all the representations of number are results of processes, are wholes generated from the elements successively. To this extent each element carries in itself a different temporal determination.

But we learned that neither the simultaneity nor the following one after the other in time enter in any way the content of the notions of plurality and thus not of the notions of number either.
As it is known already Aristotle seemed to put time and number in close relationship when he defined: time is the number of motion according to before and after. However only from Kant on it has been become more generally usual to focus on the "form of intuition" of time as the foundation for the concept of number. To be sure this happened more as a consequence of the authority of his name than of the force of his arguments. 

/30/ A serious attempt at a logical or psychological analysis of the concept of number is not found in Kant. Unity, plurality and totality constitute, according to his metaphysics, the category of quantity. Number is the transcendental schema of quantity. Kant expresses himself in detail in the *Critique of pure reason* as follows: "But the pure schema of magnitude (quantitatis), as of a concept of the intellect, is number, which is a notion which comprehends the successive addition of one to one (of the same kind); hence number is nothing else but the unity in general of the synthesis of the variety of an intuition of the same kind, by the fact that I produce time itself in the apprehension of intuition."

The passage is obscure and is not quite consistent with the explanations which Kant offers of the function of the schema. The latter themselves are not precisely uniform. Thus he says: "We want to call this formal and pure condition of sensibility, to which the intellectual concept is restricted in its use, the schema of this intellectual concept". Contrary to this, we read a few lines below: "The representation...of a general procedure of imagination, to provide to a concept its image, I call the schema for that concept".

If we transferred this latter determination to the schema of quantity then we ought to say: number is the representation of a general procedure of the imagination, [namely?] to provide to the concept of quantity its image. But by this procedure only the act of counting can be meant. Is it not however clear that 'number' and 'representation of number' are not the same? Furthermore, it is not easy to see precisely, how should we arrive, a priori, starting from the category of quantity, by means of the notion of time (as the common schema of the categories), to the individual concepts of number; and much less intelligible is the necessity that determines us to ascribe to a concrete plurality a certain and always the same number, precisely that one of which we say that it belongs to it. The theory of the schematism of the pure concepts of the intellect appears to miss, here as well as elsewhere, the purpose for which it was especially created.

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5 Kant's collected works edited by Hartenstein, III, p. 144.
6 P. 142.
We may leave aside the enumeration of the philosophers who have, after Kant, founded the concept of number upon the notion of time. Supporters of the extreme empiricism, like for instance Alexander Bain\(^7\), coincide here with those of the Kantian apriorism. From the mathematical side let two famous names be mentioned here. Sir Rowan Hamilton calls algebra straightforwardly "the science of pure time", also "the science of order in progression"\(^8\). In Germany it is H. von Helmholtz who represents the same standpoint in his essay "Ueber Zahlen und Messen"\(^9\). We will still find occasion below to deal with this essay in detail\(^10\).

Finally it should still be mentioned that most of the researchers who have put the notion of sequence instead of the notion of set as basis for the development of the number concepts as well as of the basic laws of arithmetic, were substantially influenced by the time theory.

/32/ THE COLLECTIVE AND THE SPATIAL SYNTHESIS

A. F.A. Lange's theory. While Kant viewed the notion of number as in close relationship to the notion of time, F.A. Lange thought that all that in Kant was accomplished by time, could be derived in a way by far easier and more sure from the notion of space. "Already Baumann", he says\(^11\), "has shown that number is more akin to the notion of space than to that of time... The oldest expressions for the numerals designate everywhere, as far as we know their meaning, objects in space with certain properties which correspond to number, thus for example [Viereckig] to the number four. We see from here too that number does not originally emerge for instance by systematic addition of one to one but that each of the smaller numbers, that provide the foundation for the system emerging later on, is formed by a special act of the synthesis of intuitions, with regard to which only afterwards the relations of number to one another, the possibility of adding etc. are recognized?".

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\(^8\) Cf. H. Hankel, Theorie der complexen Zahlensysteme p. 17.

\(^9\) Philosophische Aufsätze, Eduard Zeller zu seinem fünfzig-jährigen Doctor-Jubiläum gewidmet, 1887.

\(^10\) Cf the appendix to the part I of this work.

\(^11\) Logische Studien, 1877, p. 140.
"It is peculiar to the representation of space that within the large, omnicomprehensive synthesis of the multiple smaller units of the most various kinds can be distinguished with easiness and surety. Hence space is the original model not only of the continuous magnitudes but also of the discrete ones, and to the latter belongs number, whereas we can hardly think time except as continuum. Moreover, to the properties of space belong not only the relations holding among lines and surfaces of geometrical figures, but not less the relations of order and position of discrete magnitudes. If such magnitudes are considered as of the same sort /33/ and brought together by a new act of synthesis, then number as sum arises". 12

"Each number concept is obtained by us", says Lange in another place 13, "originally as the sensibly determined image of a group of objects, be they just our fingers or the [...] and [...] of a counting machine".

Our criticism will not have to look too far for points of support. The last quotation awakes a particular objection; for the well known to us concept of number appears here as an individual phenomenon, as the sensibly determined image of a group of spatial things. But there may be here just an inaccurate terminology; the idea may probably amount to saying that number is something of the sort of a sensible property observable in such groups and extractable by abstraction.

Here clearly emerges the influence of J. St. Mill, who, as we heard above 14, regards number as a sensible property which he puts on a level with color, weight, etc. But while Mill explicitly gives up a further explanation of the numerical difference, to the extent that, as it seems, he views the numerical difference as something ultimate, that is not further definable - exactly like the difference of color or weight - Lange thinks that he can show its source in the nature and properties of the notion of space.

The synthesis on which the concept of number rests (in our terminology the 'collective union') is for him a synthesis of spatial intuitions. In entirely the same way as geometry, arithmetic should rest, accordingly, on the spatial intuition. Peculiarities of the latter are supposed to be what provides the foundation of the evident truth of the axioms of

12 Logische Studien, 1877, p. 141.
13 Geschichte des Materialismus, II 26
14 Cf. the quotation on p. 11.
arithmetic and the character of absolute necessity that lies in them.\footnote{Logische Studien p. 140}

This theory surprises with the ability with which it ignores the clear witness of internal experience. Are then merely spatially divided contents countable — one will immediately ask. Don't we speak of four cardinal virtues, of two premisses of an inference? What spatial position or order is the foundation for the numerical designation in any psychic act or state when we count them?

To be sure, Lange would not be frightened by this objection; he indeed reduces not just the mathematical but all the logical thinking to the intuition of space, all the psychological is for him localized. This is not the place to subject to a comprehensive criticism this peculiar view, which appears to me as entirely untenable. Only the points that specifically concern the arithmetical problems should be highlighted.

Even if we granted the Lange basic idea, nothing more would have been proved with regard to the notion of space than before was conceded in connection with the notion of time; it would be a non-eliminable psychological pre-requisite for the genesis of the concept of number, but this it would be neither more nor less than what it is for the genesis of all other concepts.

But if spatiality belonged to all contents that we unite in thought, it would still always remain the ambiguity: to conceive contents spatially determined, and to conceive contents according to their spatial determinations.

Certainly, thereby it is not yet resolved whether the notion of space does not make, nevertheless, a particular contribution to the content of the concept of number. One sees easily that this is not the case.

Let us reflect, by means of an example, on how we unite or count collectively spatial objects. Do we all the time pay attention to the relations of position and order? Certainly not. There are infinitely many positions and orders, but the number remains unchanged. Two apples remain two apples, whether we bring them closer to each other or farther away from each other, whether we shift them towards the right or towards the left, upwards or downwards.
Number has, precisely, nothing to do with spatial position. The relations of position and order may always be implicitly connoted in the representation of a plurality of spatial objects in the phenomenon; what is certain is that they do not constitute the objects of the interest that distinguishes and determines the content of the concept of number in the uniting and counting.

But Lange not only emphasizes the spatial nature of the counted contents and their spatial relations; according to him, the individual contents must, in order for the idea of number to arise, be considered as of the same kind among each other, and be brought together as a unit by a special act of synthesis.

We leave aside the first part of the assertion; the question whether along with counting there goes necessarily hand in hand a comparing should be evaluated more exactly later on.

More important is for us the second point, namely the reference to particular acts of synthesis, by which the ideas of number come to be. Thus Lange seems to have himself felt that the "omnicomprehensive synthesis" of space, in other words that the spatial relations or combinations do not suffice to characterize the unification of the counted contents in number.

However he is so far from a clear understanding that he repeatedly /36/ refers to the idea of space as the "prototype of all synthesis" and in particular as "the prototype of the discrete magnitudes" (and number would be one of these).

The lack of clarity of the thought is already revealed by the confusion of the expressions. What is this peculiar expression "prototype" supposed to mean? Probably a sort of image-similarity must be intended. But how should a picture similarity exist between the synthesis of the "discrete magnitudes" (hence plurality and number), which consists in a unifying psychic act, and the synthesis of the idea of space, which is the union of parts of an intuition, hence a union in the content of the idea?

Lange's view becomes a bit more intelligible for us, if we consider the context in which the quoted passages are found. He touches upon the issues concerning the genesis and the content of the ideas of number with occasion of extended metaphysical considerations.
on "the extremely fruitful concept of synthesis" introduced by Kant. Some remarks on the Kantian theory of synthesis should be appropriate here, especially because they also are of interest for our aimed at characterization of the collective synthesis.

Kant uses the term synthesis (union) in a double sense: first in the sense of the unity of the parts of a whole, be it of the parts of an extension, or of the properties of a thing, or of the units of a number, etc.; secondly in the sense of the mental activity ("action of the intellect") of uniting.

It is here the case of an equivocation [...]; for the two meanings are in Kant closely related because of his view that every whole, whatever be its kind, is one brought to being out of its parts by the spontaneous activity of the mind. Hence synthesis means for him at once the uniting (the relational act) and the result of the uniting (the content of the relation).

Now, in mixing the two meanings /37/ he ends up straightforwardly referring to the union in general, even where only a union in the sense of a primary content of representation could be meant, as an "act of spontaneity", as a "[Verrichtung] of the intellect"16.

Thereby also Kant's peculiar view on the genesis of the notion of union becomes understandable. If all union exists only by virtue of uniting acts, and if its unity lies only in them, then evidently the notion of union can be obtained only by reflection on such acts.

For this is also Kant's view. To be sure, we commonly express ourselves as if the relations and unions belonged to the objects themselves, as if in relations it was a matter, not differently from absolute contents, of a passive assuming and observing. For Kant this would only be false appearance.

He says expressly: "Union does not lie in the objects and cannot be borrowed by perception and thereby first assumed by the intellect, but it is only a [Verrichtung] of the intellect, which itself is nothing else but the power to unify a priori..."17. And equally in

16 Cf. the quotation below.
17 *Critique of pure reason*, §16 of the second ed. S. W. iii vol., p. 117 of the edition Hartenstein.
another place it is said "that we cannot represent to us anything as united in the object without having first united it ourselves and among all notions the union is the only one which cannot be given by objects but can only be [verrichtet] by the subject because it is an act of spontaneity"\(^{18}\)

Lange starts from this Kantian doctrine. He has not overcome the above mentioned equivocation in the concept of synthesis, and this has undoubtedly contributed /38/ essentially to the multiple unclarities of his own presentations. Thus on the one hand he speaks of synthetic concepts, of the synthesis of the notion of space, and the like, on the other hand of synthesis as of a "creative act of our mind", of special acts of the synthesis of intuitions, which provide the ideas of number, etc.\(^{19}\). In another respect he deviates however not insignificantly from Kant's view, according to which the concept of synthesis is not obtained by analysis and abstraction from the primary contents but only in connection with the intellectual act of uniting.

Perhaps Lange started from the observation that this view stands in a certain contradiction with experience. In the majority of compound ideas we observe, if they are given to us analysed, the union of their partial contents but not the least indication of a composing activity by virtue which only that unity of content would be created.

For example, the union of the color and extension of a thing is not, and does imply either, the idea of a psychological activity but is something belonging to intuition itself and observable in it. The same holds in the relations of distance, direction etc.

Now if Lange wanted nevertheless to keep the Kantian idea that every union rests on a uniting activity, then there would only remain to assume that in such cases the uniting activity, even though we have no perception of it, is however there, that consequently it is an unconscious activity, while what it would create, /39/ the idea of union, would be in our consciousness.

\(^{18}\) Ibid. §15 p. 114

\(^{19}\) Cf. in addition to the above quotations the Geschichte des Materialismus, vol. II, p. 119-121 of the third edition. In the Logische Studien p. 136 he explains that we would have "in this expression [synthesis] in the first place not much more than a statement fixing the fact that in all our ideas one finds unity and plurality...". But a few lines below he speaks of synthesis as of a 'process', "through which we first emerge as subjects".
At any rate this is the path followed by Lange; and he pursues it to the extreme. We obtain according to him the notion of union, as well as any other notion, by analysis and abstraction starting from the primary contents. But with regard to the synthetic acts exclusively and unilaterally emphasized by Kant, by which the material ideas would first be created, they are moved into the transcendental background of life, which precedes consciousness, which is the reason why they can naturally contribute nothing to the genesis of the concept of union.

Like Kant, Lange follows also here metaphysical tendencies. Consciousness itself and with it subjectivity is supposed to originate from unconscious impressions only through a process of highest synthesis. This is how the unity of consciousness is explained. To it corresponds now as maximal synthesis of the content of consciousness the "omnicomprehensive synthesis" of the idea of space; for "space is the form of all objects" (p. 148).

As further on that supreme synthesis becomes objective in the whole of the idea of space, in the same way each individual synthetic act becomes objective in intuitive syntheses of spatial images. Thus every union is basically spatial union and relation. "In the idea of space", Lange explicitly says, "we find the intuition of? the concepts of connection and separation, of equivalence and of the relations of a whole to its parts, of a thing to its properties". Hence all logical and mathematical thinking is a thinking in spatial images; hence in particular "space is everywhere the origin of all the apriori". In fact in the apriori sciences (formal logic and mathematics) what is in question is in all cases a progress within a chain of relations connected with each other. Every axiom asserts relations among relations. If now space is form of all content also in the sense that not just no absolute content but no relational content is thinkable without having its intuitive foundation in the properties of the idea of space, in its relations and connections, then certainly the idea of space occurs in every apriori knowledge as indispensable mediator.

After all this one understands Lange's peculiar expressions, that space would be the prototype of all synthesis and in particular the prototype of numbers; one understands what a significant role is ascribed to the notion of space not just for the genesis of the notions of number but also for the entire philosophical theory of the science of arithmetic.

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20 Log. Stud. p. 148
as an apriori science, based consequently on the intuition of space.

I have deliberately gone farther than what was necessary for our immediate plans in the exposition of Lange's views. The criticism that follows, although it exclusively reflects on the issues that only interest us here, refutes Lange's theory of arithmetical knowledge as well, a theory that, as it is known, has had followers and that cannot be overlooked if we consider the further objectives of our investigations.

The theory of synthesis that we have just become acquainted with is untenable and rests on substantial misunderstandings. Kant overlooked that many unions with content are given to us where a synthetic activity creating the union with content cannot be observed. Again, Lange does not consider at all the cases where compound ideas owe their unity only and exclusively to synthetic acts, while in the primary content there is no union /41/ or is not paid attention to. According to him all unions occur in the content and namely thanks to the space form that surrounds every content. This is false.

Precisely the concepts plurality and number contradict this conception. The union of the collected contents in the plurality, of the counted objects in number is not a spatial union, as little as it can be conceived as a temporal union (and we will immediately be able to add, as little as it can be conceived as another union in the primary content).

If space is an omnicomprehensive form, then it unites not only the just counted contents but these together with all contents that are at all available. But what makes for instance the particular unity of the just formed collection of five things, by virtue of which there are precisely five of them? And whatever the connections of these five things may be - can't we immediately select just two or three or four among them by a unifying act of the interest without affecting in the least the factually available unions in content (for example distances, physical unions or the like)?

One sees that the synthesis of our concepts can lie not in the content but only in certain synthetic acts and hence can be observed only in the reflection on those acts. Lange himself must have felt this, since he assigns an important role to particular acts of the synthesis of intuitions. But how does this agree with his other doctrines? These acts, in fact, cannot be conceived as something spatial. It would be at any rate pure absurdity to accept such a view.
Thus it is clear that neither the number concepts nor the numerical relations, let alone the entire arithmetic have anything to do with the notion of space. The thesis that space is the prototype of number has no meaning, however we may want to turn it or twist it. Neither makes any sense to speak of a metaphorical equality\[?\], since on the one hand [42] there is a synthesis in the content, on the other hand there is an active synthesis, nor makes any sense to search for the intuitive foundation of the ideas and relations of number in space.

Finally we must still ask wherefrom are we supposed to get the concept of synthetic acts, with which Lange works so much, if all synthetic activity is stored in the unconscious [Jenseits], but its result, the notion of union, is placed in the primary content of consciousness, from which we simply extract it by analysis and abstraction.

Thereby it is also to be emphasized that the entire intuition that is fundamental in Lange as well as in Kant, according to which a content of relation is result of a relational act, is psychologically untenable.

The internal experience -and only this is decisive here- teaches nothing about such creative processes. Our mental activity does not make the relations; the latter are simply there, and are noticed through an appropriate direction of the interest, just as any other content\(^21\).

In the proper sense, creative acts which create any new content as a result different from them, are psychological nonsense. Certainly one distinguishes quite generally between the relational mental activity and the relation itself (between the comparing and the equality etc.). But where one speaks of such a kind of relational activity one understands thereby either the grasping of the content of the relation or the interest that selects and comprehends the relational points, which is the indispensable pre-requisite for the relations linking those contents to be noticeable. But be as it may, one will never be able to affirm that the act in question produces creatively its content.

Perhaps someone replies to us by referring to the synthetic acts /43/ whose existence we have established above in the case of the notions of number and which are, as we will see, identical with our "collective" unions. In them it would be the act alone what creates the union. – In a certain sense, this is perhaps right. The union consists only and

\(^21\) Cf Stumpf, Tonpsychologie, I, p. 104 and f.
exclusively in the unifying act itself and thus also the notion of union consists in the notion of the act. But aside from the act there is not a relational content different from the act, as its creative result, as the view opposed by us everywhere presupposes.

Only one argument we have not yet considered, namely that the oldest expressions for the numerals hint at objects in space with properties that correspond to number. It would be very daring to infer therefrom that the human intellect in all counting was limited to spatiality, once other explanations are so forthcoming. Men in the more primitive degrees of culture precisely found only occasions to count groups of spatial objects, and thus could their concept of number coincide with the concept that we now can designate only by the complex name "number of spatial objects". The more advanced culture took over the old words, whose meaning however in the meantime through metaphorical translation had extended far beyond the spatial realm /p. 44/. As most concepts, precisely also the numerical concepts have undergone a historical development.

**B. Baumann's theory.** Similar mistakes with regard to the nature of the concepts of number are found also in Baumann, by whom Lange was influenced not unsignificantly in his theory of the arithmetical concepts and knowledge. Baumann emphasizes repeatedly and vividly the role of our psychological activity in the formation of the concepts of number.

For example he says: "The union of 1, 1, 1 into three is a new act of the mind, not comprehensible to anyone who cannot perform it, i.e. the mere vision of one thing and one thing and one thing does not yet yield the number three, but this new synthesis requires being made first".

This is how the arithmetical concepts emerge, those of numbers as well as those of the operations of calculus, by means of a "spiritual activity which only in the internal

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22 The errors here [mentioned] led Lange to peculiar consequences. The origin of the categories should lie in the notion of space; its properties constitute the "norm of our functions of the intellect" etc. "Thus it is shown", Lange finally sums up (p. 149), "the notion of space with its for our intellect constitutive properties as the permanent and determining proto-form of our mental nature? , as the true objective counter-image of our transcendental ego?". As one tries to associate a clear sense with these sentences, they end up in nothing.

intuition can be generated and captured". While we thus generate in us "purely mental ideas of mathematics", on the other hand the external experience is supposed to "carry the mathematical independently from our mind and to offer it unmistakably", a circumstance by which Baumann explains the applicability of mathematics to the external world.

In particular this means with respect to number that the latter is "contained in space as much and even more than in time". "We find number again in the external world, we apply it according to our indications and it is practically validated, i.e. by the success of the calculation /45/; number is thus with space and everywhere with it; hence geometry is also reduced to arithmetical expressions". Lange has explicitly referred to this last quotation.

Thus, according to Baumann, to "the mathematical in us" corresponds a mathematical outside us, the latter being known by the former in full agreement with the ancient saying of Empedocles: "the equal is known by the equal".

As far as this theory has to do with numbers, and this is our only concern here, it is based on an erroneous conception of the abstractive process which yields these concepts.

Baumann is certainly right in strongly emphasizing the internal activity in the conception of numbers. It is beyond doubt that in the formation of numbers, as of pluralities in concreto, there is not a passive acceptance or a merely selective noticing of a content; if anywhere, there are here spontaneous activities related to the contents. According to preference and interest we can unite separated contents, from the just united separate again contents or add new ones.

A unified interest, comprehending all the contents and linking them, and at the same time with it and in it (in that mutual penetration that is peculiar to psychical acts) an act of unified conception singles out the contents, and the intentional object of this act is precisely the idea of the plurality or of the aggregate of those contents.

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24 ibid p. 669

3 ibid. p. 675

25 ibid. II, 668-71, 675

26 Log Stud. p. 140
In this manner the contents are present simultaneously and together, they are one, and by reflection on this unification of separated contents by means of that complex psychological act the general concepts of plurality and particular number originate.

If all this is true, then it is clear that Baumann starts from correct observations but ends with untenable ones. On the one hand numbers are supposed to be purely mental creations; and this is correct, insofar as numbers rest on psychical activities which we perform on the contents. In this respect we find in Baumann several correct observations and thus his view appears to be in the best harmony with ours.

But if this was the case really, if Baumann intended to claim nothing more and nothing else than that the concepts of mental activities enter in the concepts of number, how would it be possible that on the other hand he teaches a finding-again of number in the external world, a being together in and with space?

In external activities one distinguishes the activity from the work produced by it, which can continue to exist outside while the activity has long disappeared. But the psychological acts which provide a foundation for the concepts of number do not create in them new primary contents which might be found-again in space or in the external world, separated from the productive acts.

How should all the thinkable numbers, that we can count by arbitrary combining union of spatial contents, be contained in space? What is intuitively available, what we can find and notice in space, that is not numbers in themselves but only spatial objects and their spatial relations.

Thereby however no number is given yet; and where a number is given to us, the spatial syntheses are not and cannot be what brings about the unification of what is counted as such. The being together of objects in space is not yet the collective union in our mind which is essential to number. What external objects and how many of them we collect and count, this depends exclusively on our interest and thus the union of the collected is determined and completed only through a psychological act of the kind described above.

To look for numbers in space appears to be, after all this, no less absurd than to search for
judgements, acts of will, desires, and the like in space. Spatial objects are in any case the contents of these acts, not these acts themselves; spatial objects are in any case the counted objects, not the numbers. It is the same error in which Lange fell when he defined the intuition of space as the prototype of all synthesis as well as of the synthesis of numbers; the intuition of space underlying his theory of synthesis in general.

REMARK. We find number and space-intuition related in an entirely new and peculiar way in W. Brix's essay "Über den mathematischden Zahlbegriff" (Wundt, Studien, V, 671- 72). "The number of the space intuition" is according to this author a first, primitive form of the concept of number and together with the "number of the time intuition" it represents "an entirely heterogeneous and autonomous stage in the genetic development" of the concept of number.

"To be sure, number is not yet in this stage a concept...It is rather only a certain schema of perception, a sort of form of intuition in Kantian sense. For it still fully adheres to the objects of perception, that is, at this level one counts not three, four, five, but for instance three houses, four horses, etc. That is why it does not require yet any abstraction either, but simply consists, as Du Bois-Reymond puts it: "in the representation of the being separated of the objects of perception". Therefore it coincides approximately with the intuition of space, since space appears precisely as determined by the individual objects which are united in the idea of number".

Brix has taken a bit too lightly the psychology of the concept of number. What does then the little word "three" mean/48/ in the expressions: three houses, three horses, three apples etc. The idea of the objects being separated?

Well, but then this being separated, everywhere of the same kind, must have become noticed as such in itself, and abstraction is there. A general word without a conceptual basis would be a bizarre psychological and logical discovery.

One easily sees, for the rest, that the definition of the collective union of spatial objects as their being spatially separated is untenable, as it is subject to an objection which we had to raise repeatedly against the theories considered sofar.

If we extract mentally from n given spatial things, at pleasure and in arbitrary choice 2, 3, ...n-1 —I say mentally because physically we do not move them from their position—
and we count them, is there any change in the spatial intuition, say "being separated of the objects of perception" [?] (regarded by Brix as a positive moment of it)? The statement of this question suffices to make clear the untenability of the view.

And how are two, three, four spatial contents differentiated as two, three, four among each other? The being spatially separated does not admit any specific distinction at all that runs parallel to number. The nice philosophical terms 'schema of perception', 'form of intuition' only help to cover up the unclarity of the thought, and we cannot do anything at all with the special remark that this form of intuition of the spatial number approximately coincides with the intuition of space.

We want to skip the issue of how to reconcile the thesis with what follows on the same page of Brix's essay, that "number here is only a determination of the space of perception at each time".

Let a quotation however still find a place here: "The capacity of performing such conceptions [of number] must be therefore assigned to the majority of higher organisms — perhaps the conjecture that it is linked to the ability to see [! Husserl's remark] would not be entirely unjustified — for also the animals stand in position of attack? against several enemies differently than against one dog [enemy? Husserl's remark], 'also [Ente] counts its children', as Hankel emphasizes".

One sees that these venerable mathematicians make here the serious /49/ confusion between the notion of a certain set of physical individuals and the notion of their number; amazing is only that Brix follows them in this respect.

**TO COLLECT, TO COUNT AND TO DISTINGUISH**

By far more scientific and more plausible than all the theories that have been discussed so far in connection with the concepts of plurality and number, is the one to whose presentation we want to move now.

But in order that it becomes quite clear whether this theory delivers what it promises, I want to try to give to it a form as coherent as at all possible, and I rather give up linking my presentation and criticism to any of the forms in which the theory has actually occurred in this or that outstanding author. The following argument should have much
One can speak of a plurality only where objects different among each other are given. If they were all identical, then we would have no plurality of objects but precisely just one. But these diversities must also have been noticed, otherwise the various objects would constitute for our conception only an unanalysed whole, and we would find no possibility at all of arriving at the notion of a plurality.

Therefore notions of diversity belong essentially in the notion of every aggregate. While, moreover, we distinguish each individual object of the aggregate from the others, the notion of the identity of each object with itself is also necessarily given together with the notion of the distinction. Hence in the notion of a concrete plurality each individual object is thought as different from all others as well as identical to itself.

/50/ This being established, now also the genesis of the general concept of plurality is clear, as it seems. What else could be otherwise available as common to all the cases in which we speak of a plurality but those notions of diversity and identity, since, as it is known, in the abstraction of the concept of plurality one is not interested at all in the peculiarities of the individual contents?

Therefore, we obtain, starting from any concrete plurality, the general concept of plurality in that we refer, in a distinguishing mode, each content to every other content, but we consider, fully abstracting from the particular nature of the contents concretely given, each one just as something identical to itself.

In this fashion the concept of plurality emerges somehow as the empty form of diversity.

But with the concept of plurality is now immediately given also the concept of unity, and its content is to be easily seen on the basis of the preceding considerations. When we are counting in the strict sense of the word, i.e. therefore when we are performing the abstraction of number, we put each thing to be counted under the concept of unity, we consider it merely as one. Thereby nothing else should be said but precisely this: we consider each one merely as something identical to itself and different from all the others.

As distinguishing and identifying are inseparable and mutually conditioning functions, in the same way the general concepts of plurality and unity, formed by reflection on those
functions, are also interdependent, correlative concepts.

From the concept of the empty form of diversity arise, by the multiple determinations that it allows, the concepts of number. The latter are therefore nothing else but the [...] sorted out, general forms of diversity.

Thoughts like these or similar we find at work especially /51/ in the logical writings of Jevons, Sigwart, and Schuppe²⁷.

Thus Schuppe explains "that the essence of number is undefinable because it flows directly from the principle of identity. With the latter the one and the other are immediately posited, in that the one is distinguished from the other. Here is given therefore the multiplicity or plurality". "Red is not green and not blue, a is not b nor c, and b is neither a nor c, c is neither a nor b. These judgements of the simplest form are the presupposition of the predication of number, and one can, in order to express exactly the same meaning, state the number instead of the mere distinction: red, green and blue are — not for instance something — but three; one can also say in a more complete way: three diverse [things] or three diverse colors; but it is superfluous clarity...; 'three colors' says the same as 'three different colors'. What I cannot distinguish, I cannot count, it is one". — Number or the statement of multiplicity by means of a determined or undetermined numeral "asserts just diversity without naming the differences"²⁸.

Jevons in part stands even closer to the above presented theory: "Number is but another name for diversity. Exact identity is unity, and with difference arises plurality". "Plurality arises when and only when we detect difference"²⁹. /52/ Here "number" is taken, as one sees, in the broader sense, as synonymous with "plurality".

With regard to the sort of abstraction that is present here, the same author says: "There

²⁷ From the mathematical side let Paul Du Bois-Reymond be mentioned here. "Number", he says in his *Allgemeine Functionentheorie*, 1882, I, 16, "is so to speak the remainder that remains in our soul when everything that differentiates things vanishes and only the idea is kept that the things were separated". Here certainly the being different is used in the sense of being separated, which is no necessary ingredient of a theory of diversity of number. For the rest Du Bois-Reymond does not keep this thought pure; also, he seems to restrict himself to spatial objects.

²⁸ *Erkenntnistheoretische Logik* p. 405 and ff.

²⁹ *The principles of science*, 2nd. ed. (Macmillan[?]) and Co., London, 1883.)
will now be little difficulty in forming a clear notion of numerical abstraction. It consists in abstracting the character of the difference from which plurality arises, retaining merely the fact... Abstract number then is the \textit{empty form of difference}; the abstract number three asserts the existence of marks without specifying their kind". "Three sounds differ from three colours, or three riders from three horses; but they agree in respect of the variety of marks by which they can be discriminated. The symbols $1+1+1$ are thus the empty marks asserting the existence of discrimination\textsuperscript{30}.

However, in Jevons there is not a deeper psychological foundation, which we tried to produce above, mainly through a free use of Sigwart's hints.

In a first approach it looks like the theory of this kind is built on an undoubtedly sure basis, and thus the criticism might first merely be directed to the indetermination of the results.

One could regard oneself as satisfied with the thesis that plurality, or number in the wider and more indeterminate sense, is nothing else but "empty form of diversity". But this generality does not suffice to neatly characterize the content of the sharply separated concepts of number: two, three, four, etc. with respect to each other. Each of them is, after all, "empty form of diversity". What is the difference between three and two, four and three etc? Should we perhaps give the questionable answer: in two we notice one relation of distinction, in three two, in four three, etc?

\textit{P. 159}  The information given to us by the last quoted passage is obviously very meager. That "variety of marks" means either the same as number again, or it means the same as "form of diversity". But how are these forms characterized psychologically with respect to each other, so that grasped in their particular determinations, they can be clearly distinguished from one another and accordingly designated with different names as well?

Here lies an essential incompleteness of the theory; let us look into the issue of whether that incompleteness can be removed.

For simplicity let us only consider an aggregate of three objects A, B, C. In its notion must enter, according to that theory, the distinction relations

\textsuperscript{30} P. 159
A∩B, B∩C, C∩A,

(let "∩" hint at these relations); they are given together in our consciousness and perform the unification of the objects into the collective whole. Now, whatever contents of any sort at all one might substitute for A, B and C, these differences always remain available as somehow determined; they constitute therefore the 'form' of diversity, which is characteristic of number three.

However some objections arise here. If those relations of distinction are together in our conception, then, in case the basic insight of the theory is correct, should also each of those distinction notions be perceived as identical to itself and different from each other; for if for example $A∩B$ and $B∩C$ were not known as different, they would fuse together without differences, and then their foundations, as one immediately sees, could no longer appear as different from each other in the conception of the aggregate.

Hence also all the distinctions of the distinctions must be in our conception, i.e.:

$$(A∩B)∩(B∩C); (B∩C)∩(C∩A); (C∩A)∩(A∩B);$$

but the same would hold also for them, etc.

Hence in order to [obtain?] the "form of diversity" we would enter in an [...] regressus in infinitum.

But there is still a way out, through which one can avoid that consequence. If, one could reply, in distinguishing we move from A to B and from the latter to C, then a new distinction between C and A is no longer required; namely when we link to each other, by a higher act of distinction, the two distinctions $A∩B$ and $B∩C$, which are connected by the single foundation B, the possibility that A and C fuse into one is eo ipso ruled out.

Thus the following would result as the true diagram:

$$A∩B∩C$$

Whatever A, B and C might mean, this schematic picture hints at a procedure that is
everywhere of the same kind. Hence if we abstract from the peculiarities of the individual contents, retaining each one only as determined in some way, then here we have the desired form which is common to all pluralities of three contents, and because of that form we ascribe to these pluralities the number three.

In this fashion one could determine all the forms of diversity that should constitute the foundation of the application of the number designations? For instance, $A \cap B$ would be the schema of the simplest number, two; it would express that whenever a twoness is present, there is an object and still another object different from the former.

If we ascribe to a concrete aggregate the number two, then this would mean: we focus our attention merely on the fact that a content and still another content are present; [the assignment of the number two?] would rest not on the peculiarity of the distinction but on the sheer fact of such a distinction.

/55/ The schematic form for number four would be

$$\neg \neg \neg \neg A \cap B \cap C \cap D$$

and now one easily has an overview of how the forms become more complicated. There always exist distinctions which [touch each other?] (i.e. have a common foundation) and thus make it possible by means of distinguishing acts of higher order to finally be grasped in one unique act.

These schemata should serve as the images of the mental processes that they take place in the conception of any aggregate of two, three, four, etc. contents, and through reflection on these mental processes, whose well characterized diversity should be noticed internally, the number concepts would emerge.

Thus it appears that the imperfections targeted by our criticism above can be fully removed. Indeed, there is still more. The finer elaboration obtained now by the theory seems to bring about immediately as by-product of the success the solution of many questions that arise with regard to the number concepts.

Thus for instance the extraordinarily rapidly increasing complication of those forms
would make understandable for us why we reach genuine notions only of the smaller numbers, whereas we can think the greater numbers only symbolically, somehow by detours.

The independence of number from the order of the counted objects would become evident immediately by looking at the schematic form. The form remains indeed evidently unchanged, obviously, no matter how the A, B, C,... might be interchanged.

Thus several things could be said in favor of the theory. But linguistic usage seems to offer a particularly important confirmation of it. We say in the same sense: A and B are different things — are two things. Red, green and blue are three, this means that they are not for instance one but three different colors; "but that is superfluous clarity and emphasis: they are three colors says the same as three different colors". Thus Schuppe conceives also the linguistic use as in entire harmony with the distinction interpretation of the concept of number.

After all this, a well justified and coherently developed theory seems to have here a claim for being accepted. However, a more profound investigation shows that the theory is not tenable. Although no objection can be directed any longer against the coherence of the construction, the psychological foundation of the theory does not resist a sharp criticism.

It is correct that one can speak of an aggregate only where there are contents that are different from each other. Incorrect is however the additional assertion that these differences ought to be conceived as such because otherwise there would be in our conception only a non-differentiated unity without plurality.

It is important to distinguish between noticing two different contents, and noticing two contents as different from each other. In the first case we have, assuming that the contents are brought together in a unified manner at the same time, an aggregate, in the second case a distinction.

Where an aggregate is given, our conception goes just to absolute contents (namely to those that constitute the aggregate); on the contrary, where a distinction (or a complex of such) is given, our conception goes to relations among contents.
Only this is right: where a multitude of objects is perceived we hav always the right to state evident judgements, on the ground of the individual contents, which say that each one of the contents is different /57/ from each other. However, it is incorrect that we must state these judgements.

With regard to the concepts "to distinguish" and "distinction" there prevail in general several unclarities, which have originated from certain equivocations, and have contributed essentially to the errors that I consider here.

1) 'Distinction' or 'diversity' means the result of a comparison. A comparison can yield the result that the considered contents are equal, or that they are different, i.e. that they are not equal. Therefore diversity means here something negative, the mere absence of an equality. In this sense one speaks of comparing and distinguishing as correlative, closely linked activities. In fact wherever it is a matter of an arbitrary comparing, results of the two kinds emerge; affirmative judgements are stated, which acknowledge equalities, and on the other hand negative ones, which reject equalities. Now, in the use of the combination 'to compare and to distinguish' the expression 'to compare' refers to this affirming the equalities, the expression 'to distinguish' refers to the negation of the equalities.

In the case where the comparison of contents in a certain respect leads to the result of inequality, it can occur however that at least a similarity or [increase] etc is noticed; these are well characterized classes of relations in which, quite in the same way as in the case of equality, the notion of relation represents a real positive content. Now these relations were also called diversity relations, and in particular the name distinction or diversity is usual for distances in continua. In this sense we speak of distinctions of place, of time, of intensity /58/ and quality (distinction of two kinds of colors, of [sound], of odors etc).31

But now this narrower meaning of these terms led conversely again to conceive the mere cases of inequality, since they were called distinctions, as if they were primary relations [not "relations of content", errata p. 324] i.e. as in their case the relation occurred in the

31 With this it is perhaps also related that one [...] uses, especially in the case of physical contents, 'being different' in the same sense as 'being separated' (in intuition), expressions that not always coincide, as can be seen from the relation of whole and parts. Hence, we do not take this as an essentially new type of equivocation.
content of the representation, while in fact nothing else is given but an evident negative judgement, which denies the existence of such a relation (namely of an equality relation).

It may be useful anyway from the practical standpoint of the comparison to order all the results to which it can lead under the two titles of equality and diversity; however it cannot be overlooked that then under the latter rubric classes of relations coexist which are foreign to each other from the point of view of their [phenomenal?] nature, whereas in addition to this a portion of them stands in close family relationship with the relations of equality put under the other main heading.

From the psychological point of view the relations of similarity, equality, metaphysical union, etc. briefly all the relations which have the nature of representation phenomena in the narrower sense (hence primary contents, not however conceived psychic acts), belong in one class. To the latter, however, diversity in the broadest sense does not belong; for it is not a content of representation immediately observable at the same time with the foundations?, but a negative judgement, asserted on the basis of derselben?, resp? represented as asserted.

2) The name distinction is however used still in another sense, which is related to analysis. According to this, 'distinguished' means to be highlighted and particularly noticed by analysis, and 'to distinguish' amounts to 'ausscheiden', 'analyze'.

Now, in researching for the second time the conditions favoring the analysis, it turned out that a majority of partial contents is the easier isolated the greater are, in number and in degree (or distance), their differences among themselves and with regard to the environment.

These reflections, which consisted in comparisons and distinctions in the already analyzed contents, misled often into an erroneous view: as if also the process of distinguishing in the sense of analyzing were one such judging activity of distinguishing (in the sense of distinguishing compared contents). Then one inferred: in order for several contents to be able to stay in consciousness as isolated, i.e. analyzed, noticed by themselves, they must be thought as distinguished from one another, i.e. as compared and

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32 We will call them primary relations of content. More on the division here sketched, which will appear to be important for the purpose of a characterization of the collective union, in chapter III.
especially characterized according to their differences.

This is false, indeed obviously absurd. The judging activity of distinguishing presupposes evidently contents already isolated, noticed by themselves, hence these contents cannot have become observable only by the fact of having been distinguished from one another.

This is now the error committed by the theory we are criticizing when it argues: the diversities among the objects of a plurality must have been noticed as such, otherwise we would never go, in our conception, beyond an unanalyzed unit, and one could not speak of a plurality; therefore the notions of diversity /60/ must be explicitly contained in the notion of plurality.

What is right is this: if the contents were not different from each other, there would be no plurality. Moreover, it is also correct the following: the distinctions, if they were distances, must have gone beyond a certain measure; otherwise no analysis would have arisen. But it is wrong to suppose that each content becomes a particular content, i.e. noticed by itself only through the grasping of its differences with regard to other contents; whereas it is obvious that every notion of distinction presupposes as its foundation contents that are already noticed by themselves and that are in this sense distinguished contents.

In order that a concrete notion of aggregate emerge it is only required that each one of the contents comprehended in it be a content noticed by itself, an isolated content; but there is absolutely no necessity to pay attention to the distinctions of the contents, although this will happen often enough.

Something entirely analogous to what has been explained with regard to the notion of distinction holds also for identity. Both are concepts that emerge from the reflection upon certain judging activities; judging activities that in practical life have a far reaching significance and can be parallel with the representation of a manifold.

But that they do this always and necessarily, that they represent at all "constant activities, repeated in each act of thinking", in which "the self consciousness equal and identical in all acts is realized"33, and above all that the concepts of unity, plurality, determined

33 Sigwart, Logik II, 37.
number are acquired in connection with them - all that we cannot take to be established by the offered arguments.

Here as for the rest in the /61/ analysis of elementary concepts, one has been excessively subjected to the plan of regarding results of later reflections on their content as something originally contained in them or as a necessary aspect of their development.

Although the at first so convincing argument that seemed to force us with unavoidable necessity into the theory of distinction, has turned out to be insufficient, the question is not however settled of whether after all the notions of distinction are or not essential components of the concepts of number.

Sofar just the view has been refuted, according to which the conception of a manifold in concreto could not arise without the individual objects being kept separated from one another by means of distinguishing judging activities. From this it only follows that in no way a so to speak apriori necessity exists of resorting to notions of distinction in the development of the numerical concepts, and consequently to identify these concepts with those distinctions of higher order built one on the other in the guise of a pyramid.

At any rate it would be still conceivable that the direction of the interest in the formation of numbers precisely aims at the differences of the objects to be counted.

But what is decisive is the reference to the internal experience. The latter shows with complete clarity that neither the notion of a concrete plurality nor that of the corresponding number necessarily include the explicit notions of the differences between the counted individual contents.

Of differences in the sense of distances one should not think here in any case, since entirely disparate contents can be brought together and counted, whereas there are no relations of distance between disparate contents.

There would only remain, according to this, those notions of distinction as they arise from the comparison and the reflection on negative judgements. But the internal experience shows us nothing /62/ of such judging activities by which the individual contents would be conceived as different from one another when we count them; let alone a trace of those multi-level distinctions of higher order founded one on the other,
to the assumption of which the coherent development of the distinction-theory would force us.

What is certain is that we can always transform the individual contents into foundations of distinctions; but it is no less certain that the latter are not intended (meant) in counting.

To distinguish on the one hand and to bring together and count on the other are quite different activities of the mind. Only one thing and nothing more is required: that the contents to be counted be separated (i. e. noticed by themselves), but not distinguished from each other in any sense.

The friends of unconscious psychic activities may place the distinguishing acts that we deny here anyway in the nebulous domain of the unconscious; there the described 'forms of diversity' would have also their place. But I think this much is clear: those unconscious psychic mechanisms neither can have contributed anything at all to the content of our conscious notion of number, nor would be capable of providing the least explanation with regard to the genesis of these notions.

It only remains to solve the last argument that appeared to work in a particularly favorable way for a differentiation theory, namely the confirming reference to certain equivalences of the linguistic use. Red, green and blue are three colors says the same as: are three different colors; indeed the latter type of statement would involve, Schuppe believes, a superfluous clarity and emphasis. Perhaps not entirely right. One should just consider the special emphasis of the /63/ numeral, which is indispensable in order that the sense of both sentences become really equal. Red, green and blue are three colors already expresses a quite different sense. To deter a threatening confusion of several contents also the emphasis of their number can be helpful; for without diversity no number. That is an [...] function of this concept, adjusted to a quite special goal. If we say: red and green are two contrasting colors, then number has no longer that particular function. Therein lies diversity somehow, but to express it is not the special objective. And the same is shown by other examples, as many as one wishes. Saturn has three moons and two rings. This stick has a length of ten meters, etc.

It is just not right "that number only asserts diversity, without mentioning it". For the rest,

34 Cf. the quotation above.
language has special forms of numerals, in which this objective constitutes an essential part of the meaning, the so called generic numerals: of one sort [einerlei], twofold [zweierlei], etc.; it is obvious that one cannot substitute them everywhere for the basic numerals one, two, etc.

**Critical appendix.** We have mentioned among the main representatives of the difference theory, along with Jevons and Schuppe, also Sigwart. And in fact the latter has repeatedly defended the basic theses of that theory. This however should not mean that his view of the content of the number concepts entirely coincides with the above presentations.

When he says: "All the number concepts are thus only developments of the formal functions, that fulfill themselves in always higher syntheses, of the formal functions which we perform in each act of thought in general, by unit-positing and distinguishing"\(^ {35} \), one might at any rate felt warned [?] in connection with /64/ the above deduced forms of diversity.

The same can be said of another passage\(^ {36} \): "For what is posited as identical and is distinguished from another, is precisely thereby posited like this other as one; and by the fact that we in our consciousness put these correlated functions in relation to each other, with the concept of one emerges the concept of two, and with this the foundation of all number concepts".

In another place however it is spoken of a "distinction and bringing together of the acts of advancing from one to another"\(^ {37} \), of a consciousness of the transitions of consciousness\(^ {38} \), and counting itself is referred to as "the general form of the conscious advancing of a unit to the other"\(^ {39} \).

Hence in Sigwart, besides the acts of positing as identical and distinguishing, still other elements are reflected in the number concepts, not considered by us above; in his case those syntheses or compositions seem not to consist merely in distinguishing acts of higher order, as we have presupposed them to be in the coherent construction of the forms of diversity.

\(^ {35} \) Logik II, 38.
\(^ {36} \) ibid, p. 37.
\(^ {37} \) ibid. p. 38, .
\(^ {38} \) ibid. p. 41.
\(^ {39} \) ibid. p. 43.
Nevertheless he represents in many passages of his logic the essential thoughts of the difference theory. To distinguish and to posit as identical are supposed to be, according to him, activities which we perform in every conception of objects; they would be "simple, interconnected acts by which only what is many and distinguished first reaches our consciousness at all".\(^{40}\)

Another passage expresses no less clearly the same opinion\(^41\): "In order to be in consciousness several distinguished objects, distinguishing is presupposed; but at first only the result of this function enters consciousness, consists in the being one along the other of several objects, each of which is hold by itself". Here we find in full the view that we have precisely rejected.

To distinguish and to posit as identical cannot possibly have the function that Sigwart ascribes to them. Where the terms of the relation are not yet /65/ separated, where what is plural and diverse is not yet available, no relational activity and thus not either the activity of distinguishing and comparing, have any possibility at all of becoming actual.

To distinguish and to posit as identical are judging activities whose practical determination in the context of our thought seems to me lies in an entirely different direction. A is identical to itself, that is, A is not B, C, D...but precisely A. Such a reflection aims at preventing confusions of A with other contents, a goal that is attained when one seeks and highlights the "distinctions" between A and B, C, D...(i.e. the characteristic marks which belong to it and not to the others).

But while this process develops A, B, C, D....are already present to consciousness as contents separated from one another, and it is not at all its task to begin by separating what is originally a one, and thus by the isolation of the units to make the plurality first possible.

It should be clear enough from the quoted passages that precisely the equivocation lying in the expressions "distinguish" and "distinction", which associates two concepts that must be kept apart (to analyze and to distinguish by a judgement) has misled Sigwart into

\(^{40}\) ibid. p. 36.

\(^{41}\) I, 36.
his view.

It is certainly strange enough that Sigwart himself occasionally sees the right state of affairs when uncovering approximately the same error in Ulrici\textsuperscript{42}. In this regard one will be inclined to assume that Sigwart, where he speaks of distinguishing, ultimately has not a judging activity in mind but the analyzing by us opposed to the propositional distinguishing.

But I find it is not possible to defend this interpretation. Distinguishing is repeatedly put together with comparing by Sigwart, as psychological activities with reflection upon which the concepts of equality and distinction are acquired.

Thus for example we read precisely in the continuation of the last of the passages quoted on p. 64: "The notion of distinction...develops only when the distinguishing is fully conscious and one reflects on it".

It is not conceivable that here psychological activities are meant other than the judging acts in which we grasp distinctions or equalities. Surely, the analysis would not fall into consideration; for this is no psychological activity at all in the proper sense of the word. i.e. one that belongs in the domain of reflection.

Let us distinguish between a psychological event and a psychological act. Psychological acts are to conceive, to affirm, to deny, to love, to hate, to want, etc., of which the internal perception (Locke's reflection) gives us notice.

With analyzing things are quite different. Nobody can perceive internally an analysing activity. We can learn from experience that an initially unanalyzed content becomes analyzed later on; where there was one content first, now a plurality is observed. But more than this post hoc is not to be observed internally.

Of a psychological activity by which only from the unanalyzed unity plurality emerges,

\textsuperscript{42} aa0? 1\textsuperscript{i}, 279, note. "The opinion that only through a distinction a notion becomes a determined one, forgets that the distinction itself is possible only among different notions already available, and that consequently the distinction does not create the different content". Sigwart refers here to Ulrici, Compendium der Logik, 2nd. ed., p. 60.
the internal perception says nothing\textsuperscript{43}. But the fact of the performed analysis comes to our knowledge in that we compare the recollection of the unanalyzed whole with the present notion of the analyzed one.

There are kinds of acts of comparing and distinguishing which presuppose however the completed analysis.

If all this is right, then a view which puts the origin of the concepts of distinction, manifold, and number in the reflection on the activities of distinguishing in the sense of analyzing, lacks any ground.

Thus it is not a way to be followed to interpret and consolidate/67/ Sigwart's statements, and to construct a new and more defensible difference-theory of number instead of the one presented above\textsuperscript{44}.

\textsuperscript{43} Hence Stumpf in his Tonpsychologie I, 96 defines very accurately analysis as the noticing of a manifold.

\textsuperscript{44} Errors with regard to the function of distinguishing in the conception of several objects are so forthcoming that we will not be surprised to find them among earlier authors. Cf. Locke's \textit{Essay concerning human understanding} B. II, ch. XI, sect. 1, and B. IV ch. VII, sect. 4. Further, James Mill's \textit{Analysis of the phenomena of the human mind}, ed. by J. St. Mill, London 1879. II, 15: "As having a sensation, and a sensation, and knowing them, that is distinguishing them, are the same thing..."; he also repeats many times this assertion.
CHAPTER III

THE PSYCHOLOGICAL NATURE OF THE COLLECTIVE UNION

RETROSPECT

Let us look now back at our considerations so far and their results.

We took up the task of discovering the origin of the concepts of plurality and number. To this end it was necessary to take an accurate look at the concrete phenomena from which those concepts are abstracted. These were, clearly, the concrete pluralities or aggregates. Nevertheless there seemed to exist particular difficulties in the transition from them to the general concepts.

It was first of all clear that the special nature of the individual objects united in the form of a plurality could not contribute anything to the content of the corresponding general concept. The only thing that in this concept formation could come into consideration was the union [combination] of the objects in the unified representation of their aggregate.

Now, the issue was a more exact characterization of this type of union. But precisely this did not appear to be such an easy thing. In fact we became acquainted with several theories on the origin and content of the concepts plurality and number, which all failed because of misunderstandings concerning the relevant syntheses.

The first theory characterized the collective union as the mere belonging to a consciousness. It was obviously insufficient, but made us aware of an important psychological prerequisite: each collected content must be one noticed by itself.

Also it was suggested to us to view the unification of the contents as mediated by particular acts of consciousness.

Over and over again we were confirmed in this approach through the criticism of the three following theories, which were supposed to be able to construct our concepts through the "forms of intuition" of time and space. We thereby became acquainted with
time as prerequisite of number.

The last theory that we considered, the only one properly scientific, was the difference theory. It started from certain mental acts, but they were acts of distinguishing, which a deeper critique could not accept as the acts effectively playing a role for the [collectivum] and number.

**THE COLLECTION: A PARTICULAR TYPE OF UNION**

Which possibilities are now still available? We have not yet investigated all the kinds of relations - should the collective union find its place among those still waiting for our inspection? We are however released, obviously, from a detailed examination of each one of the individual kinds of relations.

Given that we know that the most heterogeneous /69/ contents can be unified in a collective manner, all relations must be left out whose domain of application is a limited one because of the nature of particular contents; hence relations of the type of similarity, [increase], continuous union etc. Indeed it seems that absolutely none of the known types of relations can satisfy the set requirements, once the temporal and the difference relations are excluded.

At most one could still think of the equality relations, for no matter how much two contents differ from each other, it will be always possible to indicate an aspect in which they are equal to each other. In fact it was also often believed (indeed the majority of scholars shares this view) that with regard to the genesis of the concepts of number one should resort to the equality relations. This should later on still [uns beschäftigen] (cf. chapter VIII).

But here it is sufficient to make the brief remark that the equalities that we in any case may discover among the collected contents, certainly cannot be the [Kitt] that brings about the synthesis of the objects hold together in the notion of aggregate; for the collection does not presuppose any comparison. When we for instance think the aggregate of time, [Tinte], and feather, we do not have to begin by comparing these concepts; on the contrary: in order to be able to do that, we must have first collected them.
Therefore it seems that nothing is left other than to claim for the collective union a new kind of relation, well distinguished from all others.
In behalf of this the internal experience also speaks positively. When we think 'together' individual contents in the form of an aggregate, this together does not admit of being analyzed into, and defined by, any other relations.

The same should be confirmed in the following considerations, which aim at characterizing still more precisely /70/ the collective union in its peculiarity vis à vis other relations.

**ON THE THEORY OF RELATIONS**

Since I am not in a position to use a firmly founded and accepted theory of relations, I am forced to insert here some general remarks concerning this very obscure chapter of the descriptive psychology.

First it will be useful to agree on the term relation. What is it that is common to all cases in which we speak of a 'relation', by reference to which precisely this name 'relation' is applied? To this J. St. Mill gives us in a note to the psychological work of his father\(^{45}\) the following intelligible and essentially sufficient answer: "Any objects, whether physical or mental, are related, or are in a relation, to one another in virtue of any complex state of consciousness into which they both enter; even if it be a no more complex state of consciousness than that of merely thinking them together. And they are related to each other in as many different ways, or in other words, they stand in as many distinct relations to one another, as there are specifically distinct states of consciousness of which they both form parts."

Let some complementary remarks be added to this. The expression "state of mind" [Bewusstseinszustand] is not to be understood here as, say, mental act, but must be taken in the broadest sense, so that it coincides in the extension of its meaning with "phenomenon". - In the statements above, Mill has, carefully seen, properly only defined the concept "to stand in [a] relation"; but what should be understood by "relation"?

That this is no [müßige] question /71/ can be already seen from the fact that Mill

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\(^{45}\) James Mill, *Analysis* II, 10
vacillates in his terminology. He repeatedly refers\textsuperscript{46} to that complex state of mind, of which he speaks above, as the "foundation of the relation" while by relation he [simply] means the relative attributes to be formed through reflection on the foundation. But it also happens\textsuperscript{47} that he says, in talking about the foundation, that it constitutes the relation, by which certainly this name becomes equivocal.

Now in order to fix our use of language we stipulate that by "relation" is to be understood that complex phenomenon which provides the basis for the formation of the relative attributes, and that by "foundation of the relation" is to be understood (in agreement with the today generally usual language) each of the related contents.

I still observe that the definition only in one respect is a bit too narrow, to the extent that it considers relations only between two foundations. But there are also relations among several foundations, and indeed also simple relations, as we shall immediately afterwards show.

For the purpose of a division of relations one could first take as guide the peculiarity of the contents which are related to each other (that is the "foundations"). However, such a division would remain in the surface. In the most diverse types of domains we find relations which have one and the same nature. Thus, equalities, similarities, etc. occur on the level of the primary contents (of the 'physical phenomena') as well as on the level of the mental acts (the 'mental phenomena')\textsuperscript{48}.

\textsuperscript{72/} But one can, and this is the deep criterion of division, classify relations according to their proper phenomenal character. From this point of view multiple divisions of relations result, among others one into the following two main groups:

1) Relations that have the character of primary contents (of 'physical phenomena' in the sense defined by F. Brentano).

Each relation rests upon 'foundations', it is a complex phenomenon that (in a sense not to

\textsuperscript{46} p. 9. Further, cf. his logic, book I, chap. 3, § 10 (Gomperz I, 56)
\textsuperscript{47} Logic, book I, ch.2, §7 (Gomperz I, 29)
\textsuperscript{48} With regard to the meaning of the terms 'physical' and 'mental' phenomenon, and to the fundamental distinction, indispensable for our subsequent considerations, which underlies those terms, cf. F. Brentano's Psychologie vom empir. Standpunkte I. Bd. 2. I. Cap.
be described more precisely) comprehends partial phenomena; but in no way every relation comprehends its foundations intentionally, i.e. in that specifically determined way in which a 'mental phenomenon' (an act of noticing, of will, etc.) comprehends its content (what is noticed, wanted, etc.).

Let us compare for instance the way in which the notion that we call similarity of two contents includes these contents themselves, with any case of intentional inexistence, and we will have to agree that we are dealing with two entirely different types of inclusion.

Because of that, precisely, similarity is not to be subsumed under the concept of 'mental phenomenon'; it rather belongs with the primary contents. The same holds of other relations too, for example equality, [increase], the continuous union (the union of the parts of a continuum), the 'metaphysical' union (the union of properties, as color, with the spatial extension), the logical inclusion (as of color in /73/ redness) etc. Each of these relations represents a particular species of primary contents (in the here adopted meaning of this term) and belongs, with reference to that, into the same principal class.

I still note explicitly, that here it is not important at all whether the foundations themselves are primary contents or any mental phenomena (conceived mental states). Also such equalities, similarities, etc. that we perceive among mental acts or states (judgements, acts of will, etc.) have in the respect here considered the character of primary contents, only that they emerge in connection with those mental phenomena and are founded in them.

Not inadequately the relations of this class qua associated with the primary contents, could be called briefly primary relations; only the misunderstanding should be avoided of thinking that thereby it is always a question of relations between primary contents, while this is not at all relevant, as emphasized above.

2) On the other hand there is a second main class of relations, which is characterized by the fact that here the relational phenomenon is a mental one. If a unified mental act is directed to several contents, then the contents are unified or related to one another with respect to that act.

If we fulfil such an act, then naturally we would in vain look for a relation or union in the conceptual content included in the act (unless there was in addition still a primary
relation). The contents precisely are united here only by the act, and therefore this union can be noticed only through a special reflection on the act.

As example any arbitrary act of conception can be put forward, judgement, feeling or will, which /74/ points to a plurality of contents. Of each of these mental acts we can say, in accordance with Mill's definition, that it puts the contents in relation to each other. Here in particular belongs for instance the above discussed difference relation in the broadest sense, where two contents are related by an evident negative judgement.

The characteristic difference of the two classes of relations can be described also by saying that the primary relations in a sense belong to the conceptual content of the same level as their foundations, the mental relations not. In conceiving the foundations, in the former case the relation is co-given immediately as aspect [Moment] of the conceptual content itself. But in the second case, the case of the mental relation, it is required for the conception of the relation first a reflective thinking upon the relating act. The immediate content of the latter is the act that establishes the relation, and only through this [it goes] to the foundations. The related contents and the relation constitute thus somehow contents of difference level49.

Another division of relations that comes into consideration for us, is the familiar one into simple and compound relations. Here often a false principle /75/ of distinction is given as criterion. Let relations between two foundations be simple, among more than two foundations be compound50. However the mere number does not justify any essential distinction, but at most it can indirectly hint at some such distinction, and indeed it should be here the simplicity and composition of relations, in the proper sense of these terms.

Really the deeper reason for taking number as the criterion of classification was contained in the view, accepted as obvious, that in the same way in which every compound relation, as a complex of relations, necessarily includes more than two

49 I have avoided, in the preceding discussions, the expression 'physical phenomenon', which in Brentano is correlated to the 'mental phenomenon', because there is some inconvenience in referring to a similarity, [increase], and the like as physical phenomenon. Also Brentano himself had in mind with this designation only the absolute primary contents and indeed individual phenomena, and not abstract aspects [Moment] of an intuition. However one sees, from the explanations above, that the mark of the intentional inexistence, which in Brentano plays the role of the first and most profound distinguishing mark of mental and physical phenomena, also in the classification of relations leads to an essential division.

50 Cf. for example Drobisch, Logik, 4th. ed., p. 34.
foundations, conversely every relation with more than two foundations is necessarily a complex of relations, namely of such that obtain between each pair of foundations.

None of this I can accept as right. On the one hand there are relations between two foundations that are compound: just consider the relation of the end-points of a series; on the other hand there are again, in the case of more than two foundations, relations that are simple, and precisely here we will have to place later on the collective union of arbitrarily many members.

For the rest also the example of the sensible equality can serve to make evident at least the possibility of relations of the latter type. An equality among more than two sensible objects we can conceive at once, under favorable and often real circumstances, without noticing in the least the great variety of simple relations that may obtain between any two of the objects\(^51\). Their number would be, already in the case of small sets (sets still properly conceivable), a number that cannot be any more controlled in one unified act. With six objects fifteen relations/76/, with seven twenty one, and so increasingly.

With this it is certainly not yet excluded that such an equality be notwithstanding a compound relation. If it is not such in the fashion of a relational complex, where the elementary relations are contained as parts noticed by themselves, it could be however in the manner of a fusion of relations\(^52\), in whose unanalyzed unity the elementary relations would be present at first as unnoticed factors. But of course it is not this. That the relation first of all appears as a simple one is in any case sufficiently proved by the possibility of simple relations with more than two foundations. One sees immediately that the view that every relation could link directly only two but not more foundations is not justified.

The most adequate definitions should be here the following: relations that are themselves again composed of relations are called compound; relations where that is not the case, are called simple.

**PSYCHOLOGICAL CHARACTERIZATION OF THE COLLECTIVE UNION**

After this digression into the theory of relations we return to the particular relations


\(^{52}\) On the concept of the fusion of relations cf. the 11th chapter.
whose characterization is our objective, and begin by asking:

Are the relations that unify the objects of the aggregate, and that we have called collective unions, primary relations in the sense made precise above, as for example the metaphysical and continuous unions, or should we perhaps put them in the class of the mental relations?

Stated more accurately: Are the collective unions in the conceptual content of the aggregate intuitively contained as partial phenomena, to be noticed especially, as for example the metaphysical unions in the metaphysical whole, /77/ or is there nothing to be noticed of such a union in the conceptual content itself, but only in the mental act which unifyingly comprehends the parts?

In order to decide this issue, let us first compare the aggregate with an arbitrary primary conceptual whole.

In order to notice in such a whole the unifying relations, analysis is required. If for example we are dealing with the conceptual whole that we call rose, then we obtain by analysis successively the different parts of the rose: the leaves, the [stem] etc. (the physical parts); then the colour, its intensity, [...] etc. (the properties). Each part is highlighted by a particular noticing and is held together with the parts already separated.

As immediate output of the analysis an aggregate results, namely the aggregate of the parts of the whole noticed by themselves.

But in addition there still appear, in connection with the unification of the parts in the intuitive whole, the union relations as particular and specific, determined primary relational contents; in our example: the continuous unions of the leaves, or the again quite differently characterized union of the properties, such as of red and of spatial extension, etc.

Thus the union relations somehow appear as the plus with respect to the mere aggregate, which [aggregate] seems to just hold the parts, but not to unify. What is therefore the difference of the case of these primary unions with respect to the case of the collective unions? Obviously this, that in the former case a unification is intuitively noticeable in the conceptual content, not however in the latter case.
The same reveals also the comparison of the collective union with the relations of equality, similarity, [increase] etc. (which form within the class of primary relations, similarly to the union relations, a psychologically well characterized group). Although they do not /78/ 'unite' the contents which serve as their foundations, they constitute nevertheless primary contents, and again the collective union, in comparison with them, appears as being somehow the case of relation-lessness. And thus one speaks also of 'non-united' or 'relation-less' contents when the point is to emphasize the absence at all of primary relations of content or of such to which the leading interest is directed. In this case the contents are precisely just thought 'together', i.e. as aggregate. In no way however are they really non-united, relation-less. On the contrary, they are united by the mental act holding them together. Only in the content of the act any observable unification is lacking\textsuperscript{53}.

Also the following circumstance shows that there is an essential distinction between the collective union and all the primary content relations, which can be explained only by the fact that the former is not to be counted at all among the primary relations.

Each relation rests on foundations and depends somehow on them. But while in all content relations the changeability of the foundations, which is admissible in order to obtain the type of the relation, is a limited one, in the collective union /79/ each foundation can be varied in a way that is fully unlimited and arbitrary, and the relation nevertheless remains the same.

The same holds too of the distinction relation in the broadest sense. Not every content can be thought as united similarly, continuously, etc. with every other one, but always as different and collectively unified. In both cases precisely the relation does not lie immediately in the phenomena themselves but is somehow external to them.

Thus many signs and above all the internal experience itself speak in favor of our having

\textsuperscript{53} Because of this J. St. Mill is entirely right when he explicitly emphasizes that objects would stand already in a relation even if we thought of them only as together. They are, precisely with respect to the mental act that thinks them together, parts of a mental whole, and can, by reflection on that, always be known as united; this constitutes their 'relation[ship]?', and only if one restricted this term to what we called primary relations, it would not be possible, naturally, to speak any longer of relation in the case of mental union. In part, to be sure, this is a terminological issue; in part however there exists de facto between content relation and mental relation so much in common, from the standpoint of the principal feature [Hauptmoment], that I do not understand why a common term should not be justified here.
to choose the second approach, according to which the collective unification is not intuitively given in the content of the representation but has its basis in certain mental acts which comprehend unifyingly the contents, a result that imposed itself to us repeatedly also in the critical discussions of the preceding chapter.

Obviously what can be here in question are only the elementary acts which are capable of comprising each and every content, no matter how disparate they might be. An attentive examination of the phenomena teaches us the following:

An aggregate emerges when a unified interest and simultaneously in and with it a unified noticing comprehends and highlights different contents by themselves. Therefore the collective union can be grasped only by reflection on the mental act by which the aggregate comes to being.

The fullest confirmation of our view is offered again by the internal experience. If we wonder, in what does the union consist when we for instance think a set of so disparate things as red, the moon, and Napoleon, we find the answer that it merely consists in the fact that we think these contents together, that we think them in one act.

For the further characterization of the collective union /80/ the following should be useful. A special mental act is required for the grasping of each one of the collected contents; putting them together needs then a new act, which obviously includes in itself those [branching] acts, and is therefore a mental act of the second order.

If a plurality is conceived through the mediation of subgroups, as when we represent a plurality of six objects in the form 3+3 or 2+2+2, the formation of each one of the subgroups requires a psychic act of the second order, so that the collective unity comprehending all of them must be established by a mental act of the third order.

How does it happen that in spite of this complication of acts directed one to the other we often prefer such a kind of conception, will draw our attention in the course of chapter XI.

Since the collective union represents a special kind of relation, it is obvious that at least the collections with two elements have the character of simple relations. But how is it with collections with more than two elements? Are they perhaps complexes or
[networks] made of collective unions between each two of their elements? I do not think so. The collecting act comprehends all elements without special collective groupings, and where we believe that we perceive them, a closer inspection shows in all cases that they are heterogeneous groupings which are on the same level with the collection.

This is what happens for instance when we perform the [gliedweise] conceptions successively, where elements are linked in the form of a series. We must abstract from this temporal link to the extent that we want to obtain a pure collective union. — It will not be possible either to defend here the view that the unity of the total collection represents a fusion, in which the elementary collections constitute at first unseparated ingredients [Moment]; for we do not find the latter even by a subsequent analysis, unless we [fulfil them afresh].

/81/ Hence we regard the collective union of arbitrarily many foundations equally as a simple relation.

The collective union plays a most significant role for our entire intellectual life. Every complicated phenomenon, which presupposes parts noticeable by themselves, every higher intellectual and emotional activity requires, in order to happen at all, collective unions of partial phenomena. It would never be possible to even reach the conception of a simpler relation (for instance of an equality, similarity, etc.) if a unified interest and with it at the same time an act of noticing would not highlight the foundations and hold them unified. This psychic relation is therefore an indispensable psychological prerequisite for every relation and union at all.

Thanks to the elementary nature of the collective union it is natural that it should have found its expression also in the ordinary language. In this respect the syncategorematic little word 'and' has been sufficient for all practical needs. By and in itself it is unimportant; but where it links two or more names it hints at the collective union of the named contents. We should not wonder why the popular language lacks a special name for the concept of collective union; to this only an [exceptional], scientific interest is directed. The [standard] purposes of thought and language precisely demand only the linguistic expression of the fact that given contents are linked in a collective fashion, and
this is achieved for our language in a fully adequate manner by the conjunction "and".\textsuperscript{54}

\textsuperscript{54} Locke describes the activity of collecting in the \textit{Essay}, B. II, ch. XI, p. 6 under the title "Compounding". He also observes that that activity also links the units of a number. However he has not seen the role that that activity plays in the abstraction of this concept [of number].