The Troubled History of Abstraction

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Jahrhundertelang wurde Abstraktion als eine Operation verstanden, durch die bei einem gegebenen Phänomen manches bewahrt, von anderem jedoch abgesehen, 'davon abstrahiert' wird. Diese Auffassung der Abstraktion ist nicht nur vom Mainstream der analytischen Philosophie und Logik als wertloser Psychologismus verworfen, sondern zudem weitgehend durch eine neue Konzeption ersetzt worden (hauptsächlich repräsentiert durch die 'Definitionen durch Abstraktion'), in der das angeblich 'psychologische' Element des 'von etwas Absehens', oder 'von etwas Abstrahirens', nicht mehr ersichtlich ist. Der Psychologismus ist dadurch natürlich überwunden worden, allerdings nur um den Preis, dass die neue Konzeption der Abstraktion mit echter Abstraktion nichts mehr zu tun hat. Das ist das 'Ärgerliche' an der Geschichte der Abstraktion.

1. Introduction

The notion of abstraction has played a central role in the history of philosophy. In section (1): What is abstraction, the authentic (in my view) meaning of "abstraction" is described. This, with related observations, constitutes a set of criteria useful in evaluating the very large literature on abstraction. Section (2) offers a brief overview of abstraction theory before the 20th century. At the end of this long history, the role played not by a philosopher, but by a mathematician: Peano, appears as prominent. However, not all is praiseworthy in Peano's theory of abstraction. While on the one hand he did better than his predecessors (or even anyone else) in the theory of abstraction, on the other hand he ended up ruining or misunderstanding his own achievement (section 3: Peano's blunder). The vicissitudes of abstraction in the 20th c. are outlined in (4): After Peano. This period is quite complex (mainly because of the proliferation of pseudo-uses of "abstraction" and cognate words, e.g. "abstract"), and its examination requires a subdivision into four parts. In a final section (5) I submit my general view on what philosophy should do about abstraction.

2. What is abstraction

1) Genuine abstraction. In the authentic meaning, abstraction is conceived as an intellectual operation which includes, as an essential feature, that in our consideration of things something is retained and something is left out (to use Locke's terms: Essay, III, 1, 9). In the 20th century the terms "abstraction", "abstract", etc. have been and continue to be spuriously used, all too often (cf. 4.1 below)
in ways that ignore, or even explicitly reject the just described essential feature, regarded as merely psychological and of no interest to logic or mathematics.

2) *Abstraction as reflected in the hiding of statements.* Normally what one retains and what one leaves out is not a linguistic entity, it is, for example, the color of a tree, the shape of a rock, the material of which a wheel is made, etc. However, if one wishes to rescue abstraction from its disastrous 20th-century history as well as from its traditional psychologistic presentation, and to put it once for all in the rigorous, logical, linguistic orbit that it deserves, it is convenient to study the abstractive process as reflected in the sentences that one admits as true or false, relative to the things under consideration. For example, in our pre-abstractive consideration of a tree we accept as true the sentence “this tree has green leaves”, but while doing abstraction, that statement becomes, so to speak, *hidden*, it disappears or is ignored – which does not mean that it is regarded as false or that one prefixes to it the symbol of negation. Focussing on sentences and language in the study of abstraction is not to be construed as nominalism (such that abstract entities end up disappearing from the philosophical scene, and only their names, as *façons de parler*, remain).

3) *Random abstraction, systematic abstraction, abstractive technique.* From the indefinitely many true statements about the tree, we have decided to hide “this tree has green leaves”. But, why this statement, why not others? Evidently, in the abstractive hiding of true sentences about objects from a given universe of discourse we may proceed in a totally arbitrary way, or we may move according to some criterion, system, or technique.

4) *The product (if any) of abstraction.* It should not be taken for granted that the operation of abstraction yields anything at all, or in other words, that the question “Does abstraction leave us empty-handed or does it yield anything at all?” has an unproblematic answer in favor of the second disjunct. Without prejudicing the various possible answers to that question, let us use, in order to refer to the alleged products of abstraction, aside from the current words “abstract” (noun) and “abstract entity”, the terms “abstractum” (singular) and “abstracta” (plural); analogously “concretum”, “concreta”.

Some philosophers cannot see anything coming out of abstraction, others grant the use of names for abstracta but do not want to commit themselves to any acceptance of abstract entities outside language (nominalism). If a philosopher wants to cross the Rubicon, moving beyond these extremely negative views and giving a denotation to the description “the product of abstraction”, then his or her most important and difficult task becomes that of exactly understanding the nature of such a product of the abstractive operation. Philosophically, it is not sufficient to content oneself with the fact that a subset of the statements regarded as true before abstraction, has now been hidden. Statements are about objects, which constitute a domain or universe of discourse. Those who believe that the description “the product of abstraction” has a denotation also must realize that the initial, pre-abstraction universe of discourse has been changed into a new one; that, to use Hermann Weyl’s words, “a new object
domain" has arisen, by abstraction, "from the original one" (ensteht aus dem ursprünglichen durch Abstraktion ein neuer Objektbereich, Weyl 1928, §2). This is easy to understand. Hiding some of the statements we take to be true (outside abstraction) obviously affects, and modifies, our view of the objects. The tree of magnificent green leaves, if we abstract from the color, is a color-less entity; the ordered pair of integers (1, 2), if we abstract from properties that are not invariant with respect to "having equal cross-products", cannot be said not to be further simplifiable, since (3, 6) has equal cross-products with (1, 2) but is simplifiable (this is Peano's example, see below; Peano says fraction, in French, instead of "ordered pair"). The difficulty lies in seeing what are, exactly, the new objects allegedly yielded by our abstraction. Indeed, the crucial philosophical question in the theory of abstraction is: What are the abstracta? What is the nature of the abstracta? Or in the metalanguage: What statements are true or false about the abstracta? This question has been clearly formulated by Paul Lorenzen:

It is possible to write a word on the blackboard, but a concept cannot be written on the blackboard. A word, as it is usually said, is concrete, whereas a concept is abstract. Whoever is critical will not be satisfied with this account, and will ask: What are the abstract objects (briefly, the abstracta)? Or more subtly: How is the term "abstract" used? (my emphasis).\(^1\)

It must be emphasized that the hardness of the question concerning the nature of the abstracta is not technical or somehow related to one’s expertise in dealing with formal systems and symbolic complications – rather, it is of a “phenomenological” type: “the difficulty lies in the phenomena, their correct description, analysis and interpretation”, to use a phrase from Husserl’s early work in the philosophy of arithmetic.\(^2\)

5) Naming the products of abstraction. Even those who deny the existence of abstracta, the nominalists with regard to abstraction, may want to make systematic use of symbols for the product of abstraction – only to say, of course, that such symbols are mere façon de parler (so, for example, Lorenzen; whether this, as it seems, inconsistent approach is tenable is another issue).

6) Abstraction performed and abstraction signified (signa, and exercit, as perhaps the scholastics would say). It may be asked how the abstractive “hiding of statements” is supposed to work – all the more because, normally, statements about nonabstracta and statements about abstracta (abstracted from the nonabstracta) are used side by side.\(^3\) It is not difficult to imagine a simple example of such

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1 Man kann ein Wort an die Tafel schreiben, aber man kann keinen Begriff an die Tafel schreiben. Ein Wort, so lautet die übliche Erklärung hierzu, ist etwas Konkretes, ein Begriff aber etwas Abstraktes. Wer kritisch ist, wird sich mit dieser Erklärung nicht zufriedengeben. Er wird fragen, was abstrakte Gegenstände (kürzer: Abstrakta) seien. Oder schon etwas gewitzter: Wie wird das Wort "Abstraktum" verwendet? (1978, p. 41, emphasis mine).

2 Die Schwierigkeit liegt in den Phaenomenen, ihrer richtigen Beschreibung, Analyse und Deutung, 1891, p. 142.

3 I am grateful to a referee for raising this question, practically reproduced here in his or her own words.
a situation. Consider the collection of fingers of my left hand, briefly FL. Then consider an abstraction which consists in hiding any statement about FL that is not true of any other collection that can be correlated one-one with FL. This abstraction yields an abstractum that we may designate as N'FL (the extreme case of the nominalist who says that this designation is merely a façon de parler is included), or more briefly as 5. I can assert as true the following conjunction: "Fortunately surgery was not required on FL after the accident & 5 is prime". This conjunction, to be asserted, does not require the psychological impossibility of simultaneously doing abstraction (to obtain the abstractum 5) and not doing abstraction (to keep the concretum FL). The abstraction for 5 is not exercita, it is merely signata.

Awareness of these six points – authentic abstraction, study of abstraction through language, abstractive techniques, nature of the abstracta (if there are abstracta), names of abstracta, abstraction performed and abstraction signified –, should help whenever one is in the process of evaluating the many authors and texts on abstraction that occur throughout the history of philosophy. When reading these innumerable texts (scholastics, Descartes, Locke, Berkeley, Cantor, Husserl ...), one should be prepared to determine 1) whether the abstraction in question is genuine, 2) whether the author in question analyzes the abstractive process with logical and linguistic tools, 3) whether the abstraction in question follows some system or technique (as opposed to being random), 4) whether the author in question at least attempts to shed some light on the nature of the abstracta, 5) whether and how the author in question names the alleged abstracta (including the case of those who, playing nominalism, have names for abstracta but reject abstracta), and finally 6) whether an abstraction is intended as exercita or as signata.

3. From Antiquity to Peano

Since Boethius launched the Latin abstractio, in connection with Aristotle's aphairessis (cf. Schneider), the term "abstraction" has been always centrally present in the history of philosophy. Abstraction has helped the Aristotelian-scholastic philosophers to claim that they could avoid, in the debate on universals, the unwanted extremes of platonism and nominalism. Abstraction is the essential ladder that allows Locke to appear as reconstructing knowledge on the basis of a cabinet of the mind that would remain, without abstraction, populated only by individual ideas or combinations thereof. In the philosophy of mathematics abstraction has played an essential, persistent role from Antiquity (cf. Lear, Cleary), the scholastics (among many, it is worth mentioning the 17th. century Caramuel, with his De abstractione arithmetica), and Descartes (Principia I, 59), to Cantor (in his famous definition of cardinality), Husserl, and Peano.

From the point of view of the logico-linguistic standards of contemporary philosophy there is much yet to be better understood in the mass of abstractive theory offered to us by the philosophical tradition, especially by the "great masters of abstraction" (as Berkeley refers, with a pejorative connotation, to the
schoolmen⁴), both with regard to the primary sources and with regard to the huge secondary literature. Two examples of questions, in the primary literature, that call for a rigorous clarification are the famous metaphor of the *intellectus agens*, that magical flashlight that illuminates the universal within the singular, and the scholastic distinction between *formal* and *total* abstraction (total abstraction is familiar: it proceeds from individuals to universals; but formal abstraction is less clear: in looking at a wooden wheel, we may leave out the matter, wood, and retain the form, circle, but is this abstractum already the geometrical entity or some shadowy approximation to it? is it already a universal or does it continue to be a particular?). One example from the secondary literature is found in the work of the neo-Thomist C. Giacon, who distinguishes, in the term “abstraction”, a “good” and a “bad” sense. The good sense is characterized as “universalization” (*universalizzare*), the bad sense as “generalization” (*generalizzare*). The difference, which seems to be of degree (universalization really catches the universals, generalization falls short of it), badly needs a precise logico-linguistic analysis.

Relative to the four evaluation standards, the situation in the Aristotelian-scholastic tradition as well as in the early modern period (Descartes, Locke, Kant repeat, albeit in watered-down form, the scholastic doctrines) is the following. 1) Abstraction seems to occur only in the genuine sense – no pseudo-abstractions, as in the 20th century. 2) There is a clear awareness that abstraction consists in the sudden “hiding” of true statements: “abstrahentium non est mendacium”: those who abstract do not lie, that is, they do not assert the contradictory of what was true before abstraction. 3) There are systematic abstractive techniques; for instance “reduplication”, that is the use of phrases like “qua”, “insofar”, which, prefixed to singular or general terms⁵, have the effect of *filtering out*, in Lear’s felicitous terminology, the predicates to be hidden (“G.W.Bush qua president of the USA” filters out, for example, the predicate “former governor of Texas” in spite of the latter being true of G.W. Bush). 4) With regard to the nature of the abstractum, not much is found: authors are not interested in describing the nature of abstracta, or perhaps they do not know how to go about it.

Surprisingly for many historians of philosophy, the best on abstraction comes from a mathematician at the end of the 19th century, Giuseppe Peano fulfills all of the above listed four conditions. 1) Peano understands “abstraction” in the genuine sense; he assigns a prominent place to the term abstraction in his vocabulary, and even relates it explicitly to the classical use of the word, citing Boethius (in 1915, *Appendice*). 2) Peano studies abstraction as reflected in statements. 3) Peano offers a novel abstractive technique. Finally, 4) Peano does something that few, if any, have dared to do, namely he describes the nature of the abstracta.

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⁴ “It were an endless, as well as useless thing, to trace the Schoolmen, *those great masters of abstraction*, through all the manifold inextricable labyrinths of error and dispute, which their doctrine of abstract natures and notions seems to have led them into” (*1964*, Introduction, 17, my emphasis).

⁵ Modern logic textbooks usually offer, as types of involved sentences that need analysis, the exceptive and exclusive propositions but omit the “reduplicative” statements, which are far more interesting, and were prominent in scholastic books.
Points (1) and (2) are obvious throughout Peano’s pieces on abstraction cited in the references. With regard to (3), the abstractive technique at which Peano hints is, in a notation different from Peano’s, the following. We start from a universe of discourse: a, b, c ... At a certain point we decide to do abstraction. Abstraction is performed (here is the novelty) relative to an equivalence relation \(x \sim y\) previously defined over a, b, c ..., and consists in retaining, in the discourse about a, b, c ..., only the statements or predicates that are \(\sim\)-invariant. The term “invariant” does not seem to occur in Peano — it occurs later on in Weyl (1928, §2) and Lorenzen — but the idea is fully clear in Peano’s texts.

With regard to (4), it must first be pointed out that Peano shows no doubts concerning the existence of a product of the mental operation called abstraction. For Peano abstraction does not leave us empty-handed, there is a product of abstraction; accordingly, Peano *names* these objects whose existence he recognizes. Suppose, as in Peano’s example, that one is moving from natural numbers and integers towards fractions and the rationals. One first considers ordered pairs of natural numbers: (1, 2), (4, 5), (3, 6) ... Next one carries out an abstraction consisting in retaining only the properties that, say, (1, 2) shares with any ordered pair \((x, y)\) such that \(x.2 = y.1\). Any other property of (1, 2) is left out. For example, “having numerator 1”, “having denominator 2”, “not being simplifiable” are all true of (1, 2) but cannot be kept, they must be hidden. This abstraction yields a new object; Peano refers to the initial concretum as \(u\), and to the abstractum as \(\varphi u\). Continuing our example, we may designate the abstractum as “\(\sim(1, 2)\)”, with the prefixed “\(\sim\)” alluding to the relation on which abstraction has been done. This new object \(\sim(1, 2)\) is different from (1, 2), since (1, 2) has quite a few properties that have been hidden and are not properties of \(\sim(1, 2)\). Peano gives as example the property of not being further simplifiable: *est une fraction irréductible*, he says in connection with his example 2/3, or (2, 3), 1894, §40. On the other hand, \(\sim(1, 2)\) is identical to, say, \(\sim(3, 6)\), although of course \((1, 2) \neq (3, 6)\). The universe of discourse — as a consequence of abstraction — has changed. This even seems to be fully acknowledged by Peano. He does not seem to say that the symbol \(\varphi u\), which he uses for the result of abstracting on the object \(u\), is only a *façon de parler*.

Now, confidently asserting the existence of the abstracta, and naming them, still falls short of telling what they are. Peano, commendably, tries to explain the nature of the abstracta as *what one obtains by considering in the initially given object all and only the properties that that object has in common with the equivalent objects*. This is far from perfect, and questions arise, for example: Is the initially given object entirely hidden, ignored, so that the abstractum is a sheer collection of properties, or is it retained, albeit confusedly, as part of the abstractum? The notation itself “\(\sim(1, 2)\)”, or Peano’s “\(\varphi u\)”, favors the second alternative, since the old ordered pair (1, 2), or Peano’s initial object \(u\), continues to be alluded to in the name of the new abstractum. (Interestingly, but not surprisingly, since abstraction and reduplication go hand in hand, the same question arises when one prefixes expressions like “qua”, “insofar as”, etc., to singular terms. Does for example the

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6 L’objet indiqué par \(\varphi u\) est donc ce qu’on obtient en considérant dans \(u\) toutes et seules les propriétés qu’il a communes avec les autres objets \(v\) tels que \(\varphi u = \varphi v\) (1894, §38).
new entity "G.W. Bush qua president of the USA" retain any link to the individual person G.W. Bush, or does it consist of sheer properties without any individual person being involved?). Aside, however, from this and other problems, Peano's description of the phenomena — recalling Husserl's phrase quoted above — is on the right track, and provides at least a temporary, tentative insight into the nature of abstracta. The abstractum is a collection of properties, namely the collection of properties that are invariant. There are of course lots of difficulties associated with what is a collection (class, set, aggregate, extension ...) and with what is a property (universal, singular, accidental, substantial ...). This is normal in philosophy. But such difficulties, while being clearly recognized, should not prevent us from confidently moving forward — at least in a practical, tentative mood — whenever we feel that we can move forward. In our particular case, we clearly see that ¬(1, 2) is like a bag which contains ¬-invariant predicates, properties or statements about (1, 2), all of them and nothing else. We know, for instance, that "x is not further simplifiable" is not in the bag, but that "x~(3, 6)" is in the bag.

One question in the theory of abstraction is whether there is a systematic abstraction that is not of the Peano type. That we can reach some practical or intuitive clarity with regard to the nature of the abstracta generated by abstraction on equivalence relations, à la Peano, does not necessarily mean, of course, that the same can be obtained in connection with the products of other abstractions.

4. Peano’s blunder

Although Peano appears to have done better than anyone else in the history of philosophy in matters of abstraction, he also managed to ruin his own achievement. In fact, there was a very serious blunder in Peano's handling of abstraction, with catastrophic consequences for the history of that notion in the 20th century.

In order to appreciate the error, it must be observed, first, that not only does a new universe of discourse {¬a, ¬b, ¬c ...} arise with abstraction, replacing the initial one {a, b, c ...}: new sentences also emerge as part of the theory developed about the new objects. One of them is the equivalence ¬a = ¬b → a ~ b. This equivalence, for those philosophers who have crossed the Rubicon (i.e. who have dared to acknowledge the abstract entity and to name it), is obviously true. While doing abstraction, if a and b are equivalent then, surely, they become indiscernible, and the product of the abstraction on a = the product of the abstraction on b. Conversely, if the products of the abstraction on a and on b are identical, then a must be equivalent to b. If a is supposed not to be equivalent to b, i.e. if it is supposed that not (a ~ b), since the predicate "not (x ~ b)" is invariant, it will be (not only true of a but even) retained under the abstraction on a whereas, obviously, that predicate is not even true of b, and would not be retained by the abstraction on b, which would contradict the assumed identity of the abstracta: ¬a = ¬b.

Peano starts well with regard to the biconditional ¬a = ¬b → a ~ b; he acknowledges its being a true sentence: "Every equality, different from identity, among
the objects of a system, is equivalent to the identity among the entities obtained from the given system by abstracting from all and only the properties that distinguish an object from its equals" (1888). Soon, however, Peano becomes unclear on the nature of the biconditional \( \neg a = \neg b \iff a \sim b \). Instead of putting forward the latter as a statement about abstraction that is plainly true, Peano confusedly speaks of the biconditional both as a definition and as an uninterpreted formula. The biconditional viewed as a definition has, in our notation, the definiens \( a \sim b \) and the definiendum \( \neg a = \neg b \). The biconditional, viewed as an uninterpreted formula has, instead of the singular terms "\( \neg a \)" and "\( \neg b \)" denoting the abstracta, expressions like "\( f(a) \)" with "\( f(x) \)" taken as a functional variable, and "\( a \)", "\( b \)" taken as nominal variables (i.e. variables for names or singular terms). All these variables are to be interpreted, and can be interpreted in indefinitely many ways, in order to obtain actual sentences. Peano lists various interpretations and ends with the following, very disappointing proclamation of philosophical neutrality or tolerance: "Which is the best among the four theories [meaning the different ways of reading the biconditional \( \neg a = \neg b \iff a \sim b \), in particular of interpreting "\( f(x) \)"] that have been presented, all of them equally logical and rigorous? The best is that which pleases most each individual instructor" (1915).

Further research should show why and how an author who had reached perfect awareness of all the abstractive background needed to view \( \neg a = \neg b \iff a \sim b \) as a true statement, ended up regarding this biconditional partly as an ex nihilo stipulation, partly as an uninterpreted formula, to which a meaning has to be given later on, according to the taste of each individual. One guesses that this was the result of philosophical confusion or weakness, not only in Peano but also in his mathematical environment (as one feels in reading, for example, the overview offered by Natucci, §§335–339).

Peano’s error has perhaps contributed to bringing about another shortcoming in his theory of abstraction. A side-effect of wrongly viewing the biconditional \( \neg a = \neg b \iff a \sim b \) as a stipulative definition is that it becomes, like all such definitions, dispensable. This naturally leads to the erroneous view that names of abstracta are always eliminable, or alternatively, that any sentence \( s \) in which a name of an abstractum "\( \sim \)" occurs can be replaced, salva veritate, by a sentence \( s^* \) which is identical to \( s \) except that "\( a \)" occurs instead of "\( \sim a \)". This seems to be implied by the following passage:

The usefulness of definitions is well known. But it should be noted that, strictly, definitions are not necessary. [...]. Every proposition about irrationals is a proposition about sets of rationals; every proposition about rational numbers is transformed into a proposition about integers, etc. (1894 §41).

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7 Ogni eguaglianza fra gli enti d’un sistema, diversa dall’identità, equivale all’identità fra gli enti che si ottengono da quelli del sistema dato astraendo da tutte e sole quelle proprietà che distinguono un ente dai suoi eguali.

8 Quale è la migliore fra le quattro teorie esposte, tutte egualmente logiche e rigorose? La migliore è quella che più piace ad ogni singolo insegnante.

9 §41. L’utilité des définitions est bien connue. Mais il faut remarquer que, à la rigueur, elles ne sont pas nécessaires. [...] Chaque proposition sur les nombres irrationnels est une proposition sur les ensembles
5. After Peano

The history of abstraction from the late 19th c. until now exhibits a variety of phenomena, to an overview of which the following sections 4.1–4.4 are devoted. First, the distinction between theory and practice of abstraction is important here. The latter badly needs theoretical clarification but continues to flourish, undisturbed by the philosophers’ disagreements or confusions (4.4). Within the theory of abstraction, the most striking phenomena are the proliferation of pseudo-uses of “abstraction” and cognate words, such as “abstract” (4.1) and even the disappearance of the term “abstraction” from the philosophical vocabulary (4.2). The theory of genuine abstraction survives, but only within a small group of scholars (4.3).

5.1 The pseudo-abstractions

Peano’s viewing the biconditional as an empty schema which anyone can interpret at pleasure, unfortunate as it was, conveniently leads us to focus on the form of the biconditional. which can be described as follows: \( f(a) = f(b) \iff a \sim b \), including: “a”, “b” as variables for names of whatever objects populate the chosen universe of discourse, \( f(x) \) as a functional sign (“the f of x”), identity, material equivalence, and “~” standing for any relation \( R(x,y) \) that is an equivalence relation. The form does not represent a logical truth: just take as universe the ordered pairs of integers, “~” as the having equal cross-products, and interpret “a” as (1, 2), “b” as (2, 4), and “f(x)” as “the numerator of x”. Under this interpretation the form \( f(a) = f(b) \iff a \sim b \) comes out false. However, the form does not always come out false. If the same interpretation is kept, except for taking “f(x)” as, for example, \( x + 1 \), the result is a true sentence. Thus, the form \( f(a) = f(b) \iff a \sim b \) is, as the manuals say, contingent.

A history of the biconditional and its instances (various interpretations of its form) should probably start with Euclid who, as pointed out by Leibniz, “since he found himself unable to define the concept of geometrical ratio absolutely, stipulated what was to be understood as equal ratios” (in H. Weyl’s words: 1928, §2). Aside from its presence in Peano, a if not the most important philosophical (disregarding mathematics) occurrence of the principle happens in the work of Frege.

At the end of §61 of Grundlagen (1884), a book devoted to answering the question “What is number?”, after having explored the whole history of the philosophy of arithmetic, Frege finds himself at a loss. Nobody else provides a satisfactory answer, but he has not much to offer either. The inventory of Frege’s knowledge about the nature of number, as of the end of §61 of Grundlagen, may be described as follows: 1) the negative knowledge that all previous accounts are unsatisfactory, 2) the practical, albeit obscure, knowledge of any professional mathematician, 3) the clearly stated and accepted thesis that num-

dé rationnels; chaque proposition sur les nombres rationnels se transforme en une proposition entre nombres entiers, etc.
bers are not sensible, physical entities, 4) the emphatic thesis that numbers are objects, not concepts, and 5) the equally clearly stated and accepted thesis that numbers are essentially linked to concepts in such a way that, without loss of generality, numbers can be referred to by means of singular terms of the form “the number of the concept F”, briefly N'F. 6) Moreover, Frege’s logicism, making of arithmetic a science no less universal than ontology (“ontologism” would be more adequate than the usual designation “logicism”), which transcends even the apriori limits of Kantian pure intuition, let alone sensible experience, clearly leads to the expectation that number turns out to be an entity not linked to any partial domain of reality or, as the scholastics would say, number is a transcendental.

In section §62, Frege realizes that there is still one more, a seventh item to be added to the inventory, namely a necessary and sufficient condition for their identity: N'F = N'G ↔ F ~ G where “~” is the relation of there being a bijection between the objects falling under the concept F and those falling under the concept G. This is not for Frege a stipulation: he obviously regards the biconditional as true. Frege’s biconditional: N'F = N'G ↔ F ~ G, is for Frege a true sentence, just as ~a = ~b ↔ a ~ b was for a time and should have remained a true statement about abstracta for Peano.

However, the addition of a seventh component to what Frege knows about the nature of number does not yet satisfy him. This is understandable. The accumulated, seven conditions do indeed narrow down the possible choices for being the denotation of the singular terms of the form “N'F”, but serious questions remain: Are there, really, possible choices? and if yes, how many? and if more than one, why should one be picked rather than another?

In such a perplexing situation, Frege must have been profoundly attracted by the following candidate for becoming denotation of the singular terms “N'F”: the extension of the concept being a concept whose objects can be one-one correlated with the objects falling under F, or, to use Frege’s own, brief terminology, the extension of the concept being a concept equinumerous with F. Extensions of concepts can be reasonably claimed as non-sensible entities, they do not appear to be linked with any partial aspect of reality, that is, they may be seen as transcendentalia, and the requirements of the biconditional are respected. Accordingly, Frege moves to stipulate that singular terms “N'F” have the just described denotation (1884, §68). Such is the final answer to the basic question of Frege’s question: What is number? If one asks, for example, What is 5?, the answer will be: 5 is the collection of all concepts whose objects can be put in one-one correlation with, say, the concept being a finger of my left hand.

To be sure, if philosophy, specifically here the philosophy of arithmetic, is committed to the difficult description of the phenomena (as in the above quoted Husserlian phrase), then it might be very hard to accept the Fregean definition. Frege, however, does not appear to have, or to be worried by such a commitment. He is just delighted to have found an object that fulfills well the requisites for being a number (he was unaware, at the time, of the paradoxes of set-theory.)

What is relevant for this paper is not the philosophical value of the final output of Frege’s definition but the method that seems to emerge behind Frege’s struggle
for reaching his goal. The method seems to have two stages: first the biconditional, secondly a nominal definition stipulating the denotation of the singular terms occurring in the biconditional. Acquaintance with Frege’s work indicates that this was not for Frege a procedure occasionally performed in the case of number. He applies it again for the definition of Wertverlauf of a function, and, as I have argued elsewhere (1982), it is plausible, and very helpful, to assume that he tacitly follows the same strategy in connection with the Bedeutung of an expression.

The method involves a dose of arbitrariness and conventionalism: this is quite obvious. Out of the many, perhaps indefinitely many candidates for becoming the denotation of the singular terms of the form “f(a)”, why any particular one should be preferred to any other, once it is shown that all of them are compatible with the biconditional, as well as with other restrictive clauses (such as, in the case of number: not being a sensible entity, being transcendental, etc.)? Frege does not answer this question, and he appears, regrettably, not to believe that an answer is needed. Again, he shows that he is not interested in the “description and analysis of the phenomena” – he is not a phenomenologist.

Frege’s student, Carnap, brought the method’s potential for frivolous and arbitrary conventionalism to a maximum. In Carnap 1956, after stating the biconditional of the first stage: f(a) = f(b) ↔ a ~ b, Carnap says that one should “look around” for suitable entities to be assigned as denotation of “f(a)”, “f(b)” ... (“suitable” meaning the same as “compatible with the biconditional”). Because of Carnap’s phrase, I have referred to this procedure as the looking-around method, or more respectfully, in Latin-English, as the circumspection method. In Quine, the method becomes an appalling exercise in frivolity (not anywhere, but in the very foundations of set theory): the choice of a denotation, in the second stage, for the “unconstrained notation” of the biconditional is guided by “convenience” or “beauty” (1963, passim, cf. also my 1979).

Surely Frege cannot be portrayed as playfully “looking around” for suitable denotations for “the number of”, “the Wertverlauf of”, or “the Bedeutung of”, and frivolously choosing what seems to be more “convenient” or more “beautiful”. It must be admitted, however, that he provided the tool for all this unfortunate looking-around conventionalism, which continued to plague 20th century philosophy till recently (one of the latest examples occurs in Dummett’s philosophy of mathematics 1991. As a related curiosity, it may be mentioned that in recent Neo-Fregeanism the looking-around method has been beheaded: the neo-Fregeans claim that the biconditional \( \text{N'F} = \text{N'G} \leftrightarrow \text{F} \sim \text{G} \) alone is sufficient, without having to look around for anything).

The purpose of this essay is not to restate my criticism of the looking-around or circumspection method \textit{per se} as philosophically worthless, but its being misnamed as “abstraction”. The origin of the misnaming lies, obviously, in Peano, or better, in Peano’s environment, where the form “f(a) = f(b) ↔ a ~ b” had lost, by Peano’s will or lack of will or both, any abstructive content but nevertheless continued to be referred to as \textit{definizione per astrazione}. The misnaming affected, or infected, not only Peano’s world but practically all the multiple manifestations of the looking-around method in the past century. There have been many
terminological variants, from Russell’s “principle of abstraction” to Dummett’s “logical abstraction” and to the neo-Fregeans’ simple “abstraction” or “abstraction principle”.

Referring to the circumspection method as “abstraction” is wrong simply because the practitioners of that method have no official plan of “retaining” or “leaving out” anything. No instructions are given ever, by the circumspect logicians, as to “hiding” (abstract from) any statements. Only benevolently, one may say that the looking-around method conceals a potential, albeit obscure and un-recognized, desire for genuine abstraction. This may be detected in the fact that any arbitrary entity\(^\text{10}\), provided it is compatible with the stipulated biconditional “\(f(a) = f(b) \iff a \sim b\)”, can be accepted as denotation of “\(f(a)\)”, “\(f(b)\)” … One may charitably notify the practitioners of the looking-around method that the truth of their endeavours (as Hegel would say) lies in genuine abstraction: the only possible therapy for the looking-around method (including its recent, neo-Fregean half-version) is that of being entirely overhauled into the form of a genuine abstraction method.

The misnaming of the looking-around method as abstraction has retroactively affected Frege: his way of defining number has been called “abstraction” or even worse, “logical abstraction” (Dummett 1991). This is wrong for two reasons. 1) It is theoretically false: the looking-around method does not include any instructions for performing any abstraction at all, and specifically in the case of Frege’s definition of number the reader is never told to “retain” and “leave out” anything. That the entities assigned by Frege to the singular terms of the form “\(N'F'\)” turn out be “abstract” entities in today’s language does not make Frege’s method into an abstraction method (besides, the mentioned use of “abstract” is one of the spurious uses of the word, quite generalized in the 20th c.: abstract is what is inaccessible to the senses). 2) Also, calling Frege’s method “abstraction” is unhistorical. Frege rejected abstraction because he saw in it a pernicious manifestation of psychologism – justifiably, in the case of number, given the lack of ability, on the part of traditional supporters of genuine abstraction, e.g. Cantor, or Husserl, or Descartes, to precisely describe the abstraction performed on, say, two cats in order to reach number 2. On one occasion, Frege came very close to genuine abstraction: when he described the output of his Axiom 5 (which is just another instance of the biconditional “\(f(a) = f(b) \iff a \sim b\)”) as the recognition of “something common” (etwas Gemeinsames, Frege 1903, §146) to the initially given objects a and b. Had Frege been aware of a better organized abstraction, where the etwas Gemeinsames is grasped by precise logico-linguistic tools rather than by magical Waschkessel strategies (cf. Frege 1894, p. 326), his view of abstraction might have changed: he might have seen a Wertverlauf or a number as genuine abstracta.

To call “abstraction” the circumspection method is the most important but not the only spurious use of the word “abstraction” in the 20th century. Another

\(^{10}\) K. Fine, interestingly, appears to have reached a perception of genuine abstraction through his reflection on the reasoning on arbitrary objects (1998), but abstraction in the spurious sense of the neo-Fregeans is what seems to end up prevailing in his work (2002).
very popular pseudo-use has to do with the introduction of free variables in a "closed" expression, or vice versa, the closure of the free variables of an "open" expression. Neither operation has anything to do with genuine abstraction; one finds, however, the former being referred to as die logistische Abstraktionstechnik in Scholz' classical abstraction monograph (1935), while the second is often called "abstraction" by leading logicians (Church's "lambda abstraction", Quine, etc.)

With regard to the adjective "abstract", especially in the context of "abstract entities", the pseudo-meaning of something that is neither tangible nor accessible to the senses, without presupposing any abstractive procedure at all, has become far more general than the spurious uses of "abstraction". One way of realizing that "abstract", in this nowadays customary sense, involves no genuine abstraction, is to observe that, in today's sense, God is, for a believer, quite abstract, whereas God is far from being a genuine abstractum.

Max Black, in his Britannica article, interestingly reveals the transition from the genuine sense of "abstraction" to the spurious uses, and how philosophers in the logical and analytic tradition, in the 20th century, definitely saw the spurious uses as an improvement upon what was for them the "too vague" genuine sense. Black's last statement, in his article, is worth quoting: "The above technical devices [i.e. the spurious uses] can be regarded as instruments for sharpening the vague notions [i.e. genuine abstraction] discussed at the beginning of this article".

If one looks for an explanation of the pseudo-uses of "abstraction", obviously one reason is the strange fact that in Peano's circle the phrase "definizione per astrazione" was kept while abstraction itself was removed. This should be considered as the primary cause. Frege's negative view of what he found, in Cantor and others, on abstraction has certainly been influential too.

5.2 The disappearance of the word "abstraction"

While the terminology of abstraction continued to be used in the century after Peano, and continues to be used by many authors today, although in a spurious sense, there is also a recent trend towards dispensing with abstraction altogether (as Russell would put it, 1956, p. 326). The disappearance of the term "abstraction" from the foreground of philosophical terminology can be observed at various levels, ranging from "companions" to special areas of philosophy (e.g. Sosa and Dancy, Sosa and Kim) to general encyclopedias of philosophy (e.g. Routledge) and is even reflected in general reference works such as the Encyclopedia Britannica: while the 11th ed. still offers an entry for the genuine meaning of "abstraction", the latest, 15th ed. does not have it anymore (the word "abstraction" occurs only as part of an entry: the compound French expression: "abstraction – création", which has to do with art).

The Stanford encyclopedia of philosophy does not have an article "abstraction" and does not even mention it as a "projected entry", but already includes an article on "abstract objects" (Rosen) where, on the one hand, genuine abstraction

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11 For Church, cf. his contribution to the article on abstraction in Runes; Quine's pseudo-use is found in most of his works. Cf. also my 1991.
is discarded as something "wedded to an outmoded philosophy of mind", while on the other hand, spurious uses abound, particularly in connection with Dummett and the neo-Fregeans.

5.3 Secluded survival of the theory of genuine abstraction

After the wreck of genuine abstraction around 1900 it becomes very difficult to find, especially in the world of logic, foundational studies, and more generally of analytic philosophy, any supporters of genuine abstraction. Hermann Weyl is a rare case. In the 1920s, Weyl uses Peano's phrase "definition by abstraction" in the original, intended sense (1928, §2). Later on Paul Lorenzen emerges with an energetic defense of genuine abstraction in his 1955, §10. Lorenzen appears to be the first to state that genuine abstraction had been lost in the Frege-Russell tradition, as well as to denounce the usurpation, that is, the fact that the terminology of abstraction had become a terminology to designate the looking-around method (1955). The emphasis placed by Lorenzen on genuine abstraction was transmitted to his students (the "Erlangen school") and to those who studied his work. Lorenzen, in nominalist key, thought of the abstracta as just names, and names that are entirely eliminable. Everything said about abstracta is a mere façon de parler that can also be said of the concreta initially given. Such a nominalism is, to be sure, a way of avoiding the question "What can be said truly about the abstracta?", or equivalently, the question on the nature of the abstracta. In fact, the criticisms against Lorenzen's abstraction (Siegwart) hinge on that issue.

Not everything chronologically "after Peano" in the theory of authentic abstraction is related to Peano. Genuine abstraction theory occurs in neo-scholasticism, Marxism, Piaget's "genetical epistemology", and some other philosophical schools developing outside the Peano genealogical tree. In all these cases it is, simply, the classical, genuine abstraction that continues to be mentioned or studied.

5.4 The continuing practice of genuine abstraction

Quite regardless of theoretical discussions on abstraction as well as of the proliferation of pseudo-uses of the term, the fact of the matter is that genuine abstraction continues to be performed and practised – with the limitations, naturally, of things done in the dark, without theoretical awareness and without the appropriate conceptual framework and support. There are simple ways to see to what a great extent authentic abstraction is a component of so many human activities and disciplines. Searching the subject "abstraction" in the catalogue of an important library will probably generate a large number of entries. Browsing the index of an important general reference work such as the Encyclopedia Britannica will lead to the discovery of a surprising variety of examples of genuine abstraction being actively performed in such diverse fields as education, mathematics, art, and even meditation (while, paradoxically, as noted above, the same Britannica, in its latest editions, lacks a special entry for the word "abstraction").

Let us consider, as an example of the continuing practice of genuine abstraction in the 20th century, the small and popular introductory manual to set theory by
Kamke, which has nurtured generations of students in German-speaking countries, as well as in many other cultural domains. I would like to argue that the prestigious mathematician, in his elementary book, does not follow the looking-around method, à la Quine, but rather continues to apply Cantor's genuine abstraction approach in the foundations of set theory, started by Cantor, albeit in the same defective way as the founder of set theory, and even without using, as it seems, the term "abstraction".

Kamke defines the cardinal number of a set in the following way, presupposing the notion of equivalence, i.e. bijective correspondence between sets:

By cardinal number of power m we understand an arbitrary representative M of a class of sets mutually equivalent. The cardinal number or power of a set M will we designated by |M|. The notation |M|, accordingly, simply means that the set M can be replaced by any set equivalent to it; we have |M| = |N| for M ~ N. (Ch. II, § 8).

There is a collection of objects (sets): M, N ... provided with an equivalence relation (bijection between two sets). There are no indications that Kamke wants the reader to go through the two-stages procedure of first accepting the biconditional \(|M| = |N|\) iff \(M \sim N\) and then looking around for a convenient nominal definition, whereby the singular terms "|M|", "|N|" receive a denotation. In this respect, the phrase "an arbitrary representative M of a class of sets mutually equivalent" may be misleading. The word "arbitrary" naturally reminds us of the arbitrary choices to be performed in the second stage of the looking-around method. In my view, the meaning of Kamke's "arbitrary" is very different. It repeats an approach notoriously found already in Berkeley's rejection of abstract entities. Berkeley claims that there are no abstract entities in addition to the individuals, no abstract triangle beyond this or that individual triangle. He wants to keep, however, universals, or general entities, for example the general idea of triangle. He believes this can be achieved without leaving the domain of the individuals, and simply by taking one, any of the individual triangles as a "representative" of all triangles. The universal triangle is this arbitrary individual triangle ... qua triangle. The question arises, however, of what is the denotation of the phrase "this-triangle-qua-triangle". Obviously it is the output of an abstraction, an abstractum. Berkeley does not realize that his "representative" conceals the very abstraction which he abhors. Genuine abstraction, not the looking-around method, is fully at work, behind Kamke's "arbitrary representative" as well as in Berkeley. The last clause (crucial, as well as deceivingly brief) of Kamke's quoted passage, means the reader who intends to learn about cardinal numbers will have to ignore (leave out, abstract from), in his or her discourse about a set M, anything that is not also true of any other set \(M^*\) such that \(M \sim M^*\), where \(\sim\) denotes the bijection between the two sets. The reader of Kamke, if really willing to understand what set-theory is about, will have to pay attention, in his or her dealing with a set, only to the invariant predicates. No signs that the reader of Kamke has to abide by "beauty" or "convenience" criteria as in the case of the student of Quine's 1963.

\[12\] These remarks were prompted by the insightful comments of a referee.
While it seems clear to me that Kamke's manual is immune to the looking-around that was already in the air when the author was writing it, at the same time it must be acknowledged that from the point of view of the theory of abstraction the shortcomings are striking. Two of them are worth mentioning: 1) the failure to enlighten the reader with respect to the nature of the abstracta; 2) the failure to face the issue of talking about abstracta.

The first shortcoming has been discussed above: the student of Kamke's manual wrongly believes that nothing new arises when he or she is asked to consider an arbitrary member of the initially assumed sets M, N ... To be sure, there is a new notation: |M|, |N| ... but only the exceptionally demanding and philosophically oriented reader may want to ask for the rationale of such an additional notation. As in the case of Cantor's similar notations, the new symbol hints at the appearance of a new entity: an abstractum, which is not any arbitrary member of the initial collection M, N .... The nature of this abstractum should be explained but it is not.

The second theoretical gap in our author (an excellent mathematician who practises abstraction but does not do a good theory of it), is that in his book there are statements or predicates A(x) that are not eliminable, that is, that are said truly of the abstractum but are false of the corresponding concretum. If I assert, truly, of |M| a predicate that is →-invariant: A(|M|), and A(x) es eliminable, I will be also able to drop the vertical bars around M, and say, truly too, A(M). But the predicate "x is represented by M", said of the cardinal |M|, does not allow us to delete the vertical bars: the word "represented" should be re-defined, as a relation among sets, and not as a relation from the cardinal to the sets that represent it (in the original sense). There are examples more complicated, such as "We must show that the product of cardinal numbers is independent of their particular representatives" (II, §9), or when Kamke quantifies relative to the cardinals, or states an identity between two cardinals. Of the latter there is a nice example precisely towards the end of the quoted passage: "|M| = |N| for M ~ N". Let M = the set of fingers of my left hand, N = the set of fingers of my right hand. Deleting the vertical bars - eliminating the abstract terms - would lead to a falsity: M = N.

6. Conclusion

In the past century, and in recent times, three main classes of philosophers can be distinguished with respect to abstraction: 1) those who continue to use "abstraction" and related terms in the genuine, historical sense, 2) those who use the same words in spurious senses, 3) those who do not even include the word abstraction in their technical terminology. Thus, the question "What should the philosophy of the 21st century do about abstraction?", quite natural at the end of the troubled story, is ambiguous: the answer depends on which one of the three main classes is being referred to.

Members of group (1) have one fundamental task to complete: the clarification of the nature of abstracta, beyond the tentative, practical limits reached in connection with the Peano-style of abstraction. If this task is viable, and successful,
that is, if the existence of abstracta and their nature can be satisfactorily clarified, the theory of abstraction will have to regain its place within logic and philosophy, as had been the case for centuries, although in an upgraded – logically and linguistically manageable – way. If supporters of genuine abstraction fail in their fundamental task, they should simply join group (3).

With regard to members of group (2), they should remove the word “abstraction” and related expressions from their technical vocabulary. Of course names can be imposed at pleasure – but especially in the case of “abstraction” (more than with “abstract”) the misnaming is far from innocent: it deceives others, as well as, in the first place, those who do the misnaming. This is because the word “abstraction” suggests that there is some secret, awesome operation going on behind the scenes – behind the symbolic operation of punching out a singular term from a sentence (Scholz) or of closing a free variable (Quine and many others), or behind the empty arbitrariness of the looking-around method (Dummett), or behind the incurable sterility of any biconditional of the form of “f(a) = f(b) ↔ a ∼ b” (neo-Fregean’s Humé’s principle).

In sum, unless genuine abstraction can be vindicated (essentially meaning: unless the nature of the abstracta can be clarified), all philosophers should join group (3), and “abstraction”, “abstract” …, should cease to be *termini technici* in philosophy. This, to be sure, would not be a happy ending, in view of the continuing existence of the *practice* of abstraction in the most diverse disciplines, arts, and technologies. The deletion of abstraction from the list of philosophical topics would amount to the admission that philosophy (specifically, epistemology and metaphysics) is incapable of understanding one important feature of knowledge and reality.

7. References


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