Linear Quadratic Optimization for Models with Rational Expectations[1]

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Abstract

In this paper we present a method for using rational expectations in a linear-quadratic optimization framework. Following the approach put forward by Sims, we solve the model through a QZ decomposition, which is generally easier to implement than the more widely used method of Blanchard and Kahn.

1. Introduction

Starting in the late 1970's there was much criticism of the use of control theory in economics. In this view one of the major drawbacks was that it could not deal with rational expectations. Subsequently a number of generic methods to solve models with rational expectations (RE) were developed. For instance, Fair and Taylor (1983) use an iterative method for solving RE models and, in the tradition of Theil (1968), Fisher, Holly and Hughes Hallett (1986) use a method based on stacking the model variables. A hybrid method based on the saddlepoint property is presented in Anderson and Moore (1985).

The Blanchard and Kahn method (1980) is another well known method for solving linear models with RE in discrete time. By decomposing the model into stable and unstable parts, the unstable part can be solved backward in time and the stable part forward in time. Although the BK approach is theoretically a powerful method, for practical implementation it has some serious drawbacks. First, not all linear RE models can be put into first order linear form as required by the Blanchard and Kahn method, and second, the method is based on the *Jordan canonical form*. The procedure to put the model into the Jordan canonical is not widely available in software libraries and is known to be numerically unstable, see Moler (1994).

In recent work, Amman, Kendrick and Achath (1995), Amman (1996), we presented a procedure that introduces RE in a linear-quadratic (LQ) control framework based on the Blanchard and Kahn method. Due to the limitations of the Blanchard and Kahn approach and the fact that we had to rely on the diagonalization of the transition matrix, this work could only deal with a limited set of models. Recently, Sims (1996) proposed a different method for solving linear models with RE allowing for a broader range of models. This method is not based on the Jordan canonical form, but uses the more widely available *QZ* form that is based on generalized eigenvalues and which is more numerically stable. In this paper we will follow the paper of Sims and incorporate his

approach into the LQ framework.

2. Problem statement and solution

Following Kendrick (1981), the standard single-agent linear-quadratic optimization problem is written as:

Find the set of admissible instruments $U = \{u_0, u_1, \ldots, u_{T-1}\}$ that minimizes the welfare loss function

$$J_T = \beta^T L_T(x_T) + \sum_{t=0}^{T-1} \beta^t L_t(x_t, u_t) \tag{1}$$

with

$$L_T = \frac{1}{2}(x_T - x_T^*)' W_T (x_T - x^*)$$

$$L_t = \frac{1}{2}(x_t - x_t^*)' W_t (x_t - x_t^*) + \frac{1}{2}(u_t - u_t^*)' R_t (u_t - u_t^*) + (x_t - x_t^*)' F_t (u_t - u_t^*)$$

subject to the model

$$x_{t+1} = A_t x_t + B_t u_t + C_t z_t \tag{2}$$

The vector $x_t \in \Re^n$ is the state of the economy at time t and the vector $u_t \in \Re^m$ contains the policy instruments. The initial state of the economy x_0 is known, x_t^* and u_t^* are target values. W_t, R_t and F_t are penalty matrices of conformable size, β being a discount factor.

The above model is straightforward to solve and there are a number of packages available. However, a serious drawback for economics is that equation (2) does not allow for rational expectations. One way of allowing RE to enter the model is to augment equation (2) in the following fashion

$$x_{t+1} = A_t x_t + B_t u_t + C_t z_t + \sum_{j=1}^{k} D_{j;t} E_t x_{t+j} + \epsilon_t \tag{2a}$$

where the matrix $D_{j;t}$ is a parameter matrix, $E_t x_{t+1}$ is the expected state for time $t+j$ at time t, k the maximum lead in the expectations formation and ϵ_t is a white noise vector. In order to compute the admissible set of instruments we have to eliminate the rational expectations from the model. In a previous paper Amman, Kendrick and Achath (1995), we used the Blanchard and Kahn method to solve the RE in the model. However, as mentioned above, the Blanchard and Kahn method has some features that impede practical implementation. Sims (1996) proposes a different route by applying a *generalized eigenvalue* approach. In order to apply Sims' method we first put equation (2a) in the form [2]

$$\Gamma_{0;t} x_{t+1} = \Gamma_{1;t} x_t + \Gamma_{2;t} u_t + \Gamma_{3;t} z_t + \Gamma_4 \epsilon_t \tag{3}$$

where

$$\Gamma_{0;t} = \begin{bmatrix} I & -D_{1;t} & -D_{2;t} & \cdots & -D_{k-1;t} & -D_{k;t} \\ I & 0 & 0 & \cdots & 0 & 0 \\ 0 & I & 0 & \cdots & 0 & 0 \\ \vdots & & & \ddots & 0 & 0 \\ 0 & & \cdots & & I & 0 \end{bmatrix}$$

$$\Gamma_{1;t} = \begin{bmatrix} A_t & 0 & 0 & \cdots & 0 \\ 0 & I & 0 & \cdots & 0 \\ 0 & 0 & I & & 0 \\ \vdots & \vdots & & \ddots & \\ 0 & 0 & 0 & & I \end{bmatrix} \quad \Gamma_{2;t} = \begin{bmatrix} B_t \\ 0 \\ \vdots \\ 0 \end{bmatrix} \quad \Gamma_{3;t} = \begin{bmatrix} C_t \\ 0 \\ \vdots \\ 0 \end{bmatrix} \quad \Gamma_4 = \begin{bmatrix} I \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

[2] Note that in contrast to Sims (1996) the variable z_t contains exogenous variables and not random variables. Hence, the matrix Ψ in Sims' paper is set to zero.

and the augmented state vector

$$\tilde{x}_t = \begin{bmatrix} x_t \\ E\, x_{t+1} \\ E\, x_{t+2} \\ \vdots \\ E\, x_{t+k-1} \end{bmatrix} \tag{4}$$

Taking the generalized eigenvalues of equation (3) allows us to decompose the system matrices $\Gamma_{0;t}$ and $\Gamma_{1;t}$ in the following manner, *viz.* Coleman and Van Loan (1988) or Moler and Stewart (1973) ,

$$\Lambda_t = Q_t \Gamma_{0;t} Z_t$$

$$\Psi_t = Q_t \Gamma_{1;t} Z_t$$

with $Z_t^{\prime} Z_t = I$ and $Q_t^{\prime} Q_t = I$. The matrices Λ_t and Ψ_t are upper triangular matrices and the generalized eigen values are $\forall i \; \lambda_{i;i} = \psi_{i;i}/\lambda_{i;i}$. If we use the transformation $w_t = Z_t^{\prime} x_t$ and $w_{t+1} = Z_t^{\prime} x_{t+1}$ we can write equation (3) as

$$\Lambda_t w_{t+1} = \Psi_t w_t + Q_t \Gamma_{2;t} u_t + Q_t \Gamma_{3;t} z_t + Q_t \Gamma_4 \varepsilon_t \tag{5}$$

It is possible to reorder the matrices Z_t and Q_t in such a fashion that the diagonal elements of the matrices Λ_t and Ψ_t contain the generalized eigenvalues in ascending order. In that case we can write equation (4) as follows

$$\begin{bmatrix} \Lambda_{11;t} & \Lambda_{12;t} \\ 0 & \Lambda_{22;t} \end{bmatrix} \begin{bmatrix} w_{1;t+1} \\ w_{2;t+1} \end{bmatrix} = \begin{bmatrix} \Psi_{11;t} & \Psi_{12;t} \\ 0 & \Psi_{22;t} \end{bmatrix} \begin{bmatrix} w_{1;t} \\ w_{2;t} \end{bmatrix} + \begin{bmatrix} Q_{1;t} \\ Q_{2;t} \end{bmatrix} \Gamma_{2;t} u_t + \begin{bmatrix} Q_{1;t} \\ Q_{2;t} \end{bmatrix} \Gamma_{3;t} z_t + \begin{bmatrix} Q_{1;t} \\ Q_{2;t} \end{bmatrix} \Gamma_4 \varepsilon_t \tag{6}$$

where the unstable eigen values are in lower right corner, that is the matrices $\Lambda_{22;t}$ and $\Psi_{22;t}$. By forward propagation and taking expectations, it is possible to derive $w_{2;t}$ as a function of future instruments and exogenous variables, Sims (1996, page 5)

$$\circ_t = w_{2;t} = i \sum_{j=0}^{\infty} \tilde{M}_{t;j} - {}_{22;t+j}^{i1} Q_{2;t+j} \left(i_{2;t+j} u_{t+j} + i_{3;t+j} z_{t+j} \right) \tag{7}$$

The matrix $\tilde{M}_{t;j}$ is defined as

$$\tilde{M}_{t;j} = \prod_{i=0}^{j-1} M_{t+i} \quad \text{for} \quad j > 0$$

and

$$\tilde{M}_{t;j} = I \quad \text{for} \quad j = 0$$

with

$$M_t = - {}_{22;t}^{i1} \alpha_{22;t}$$

Given the fact that $- {}_{22;t}$ contains the eigenvalues outside the unit circle, we have applied the follow condition in deriving equation (7)

$$\lim_{j \to \infty} \tilde{M}_{t;j} = 0$$

In contrast to Sims, M_t is not time invariant since we explicity want to allow for time dependent matrices in the model[3]. Reinserting equation (7) into equation (6) gives us

$$\tilde{\alpha}_t w_{t+1} = \tilde{-}_t w_t + \tilde{i}_{2;t} u_t + \tilde{i}_{3;t} z_t + \tilde{i}_{4;t} \varepsilon_t + \tilde{\circ}_t \tag{8}$$

with

$$\tilde{\alpha} = \begin{pmatrix} \alpha_{11;t} & \alpha_{12;t} \\ 0 & I \end{pmatrix} \quad \tilde{-} = \begin{pmatrix} -_{11;t} & -_{12;t} \\ 0 & 0 \end{pmatrix} \quad \tilde{i}_{2;t} = \begin{pmatrix} Q_{1;t} \\ 0 \end{pmatrix} i_{2;t}$$

$$\tilde{i}_{3;t} = \begin{pmatrix} Q_{1;t} \\ 0 \end{pmatrix} i_{3;t} \quad \tilde{i}_{4;t} = \begin{pmatrix} Q_{1;t} \\ 0 \end{pmatrix} i_4 \quad \tilde{\circ}_t = \begin{pmatrix} 0 \\ \circ_t \end{pmatrix}$$

Knowing that $x_t = Z_t w_t$ and $x_{t+1} = Z_t w_{t+1}$ we can write equation (8) as

[3]As Lucas (1976) pointed out the parameters in the model might change due to policy announcements. Hence we would like our method to be able to deal with changing parameters in the system matrices.

$$x_{t+1} = \tilde{A}_t x_t + \tilde{B}_t u_t + \tilde{C}_t z_t + \varepsilon_t \tag{9}$$

with

$$\tilde{A}_t = Z_t \Phi_t^{-1} \tilde{\Gamma}_t Z_t^0 \quad \tilde{B}_t = Z_t \Phi_t^{-1} \tilde{\gamma}_{2;t} \quad \tilde{C}_t = \begin{bmatrix} Z_t \Phi_t^{-1} \tilde{\gamma}_{3;t} & Z_t \Phi_t^{-1} \end{bmatrix} \tag{10}$$

and

$$\Phi_t^{-1} = \begin{bmatrix} \Phi_{11;t}^{-1} & -\Phi_{11;t}^{-1} \Phi_{12;t} \\ 0 & I \end{bmatrix} \quad z_t = \begin{bmatrix} z_t \\ \xi_t \end{bmatrix} \quad \varepsilon_t = Z_t \Phi_t^{-1} \tilde{\gamma}_{4;t} \varepsilon_t \tag{11}$$

In non-pathological cases, Stewart (1973), the matrix $\Phi_{11;t}$ will be nonsingular. With equation (9) we have transformed equation (2a) into the form required by equation (2) enabling us to set up an iterative scheme using the LQ framework in equations (1)-(2). Knowing that θ_t depends on the future the basic algorithm works like this [4]

Step 0. Set the iteration count $\nu = 0$ and set the instruments u_t^{ν}, $t = \{0;1;\ldots;T+s-1\}$, set x_0 (see below).

Step 1. Compute θ_t^{ν}, $t = \{0;1;\ldots;T+s-1\}$, by using

$$\theta_t^{\nu} = -\sum_{j=0}^{s-1} \tilde{M}_{t;j} - \Phi_{22;t+j}^{-1} Q_{2;t+j} \left(\gamma_{2;t+j} u_{t+j}^{\nu} + \gamma_{3;t+j} z_{t+j} \right)$$

Step 2. Apply the LQ framework in equations (1)-(2) to compute a new set of optimal instruments $u_t^{\nu+1}$ using the equation below in place of equation (2)

$$x_{t+1}^{\nu+1} = \tilde{A}_t x_t^{\nu+1} + \tilde{B}_t u_t^{\nu+1} + \tilde{C}_t z_t^{\nu}$$

Step 3. Set $\nu = \nu + 1$ and go to Step 1 until convergence has been reached.

[4] The Matlab implementation of this algorithm can be obtained through the corresponding author.

There is still one remaining issue we have to take care of. In order to apply the algorithm for $^{o} = 0$ we need to have an initial value of the augmented state vector, which is

$$\tilde{x}_0 = \begin{bmatrix} x_0 \\ E\, x_1 \\ E\, x_2 \\ \vdots \\ E\, x_{k\,i\,1} \end{bmatrix} \tag{12}$$

However, at the beginning of the first iteration step the expectational variables are not known, so we have to give the expectational part of $\tilde{x}_0^0$ an arbitrary value. Once we have completed the first iterations we can the computed states to reinitialize $\tilde{x}_0^1$. By iterating on the expectations the above algorithm produces a expectation consistent solution of the model formulated in equations (1) and (2a)[5]. In the next section we will apply this algorithm using a simple macro model.

3. An example

Consider a simple macro model with output, x_t, consumption, c_t, investment, i_t, government expenditures, g_t, and taxes $¿_t$. The problem can then be stated as:

Find for the model

$$x_{t+1} = c_{t+1} + i_{t+1} + g_{t+1} \tag{13}$$

$$c_{t+1} = 0.8\,(x_t \;i\; ¿_t) + 200 \tag{14}$$

$$i_{t+1} = 0.2E_t x_{t+2} + 100 + ²_t \tag{15}$$

$$g_{t+1} = u_t \tag{16}$$

$$¿_{t+1} = 0.25\, x_{t+1} \tag{17}$$

[5]It is beyond the scope of this paper to address fully the issue of time-consistency. The above algorithm produces a time consistent, but not necessarily, fully optimal solution of the model. See Kydland and Prescott (1977) or Hall and Henry (1988).

with $x_0 = 1500$, a set of admissible control $U = \{u_0; u_1; \ldots; u_9\}$ to minimize the welfare loss function

$$J_T = \frac{1}{2}(x_{10} - 1600)^2 + \frac{1}{2}\sum_{t=0}^{9}\{(x_t - 1600)^2 + \gamma_t^2\} \tag{18}$$

If we reduce the above model to one equation for output we get

$$x_{t+1} = 0.6x_t + u_t + 0.2E_t x_{t+2} + 300 + \varepsilon_t \tag{19}$$

which leads to the augmented system

$$\begin{bmatrix} 1 & -0.2 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x_{t+1} \\ E_t x_{t+2} \end{bmatrix} = \begin{bmatrix} 0.6 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x_t \\ x_{t+1} \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u_t + \begin{bmatrix} 300 \\ 0 \end{bmatrix} z_t + \begin{bmatrix} \varepsilon_t \\ 0 \end{bmatrix} \tag{20}$$

where $z_t = 1 \ \forall t$. Applying the QZ factorization, Coleman and van Loan (1988), to compute the generalized eigen values of the model gives us the time invariant solution

$$\Lambda = \begin{bmatrix} 1.0822 & -0.9136 \\ 0 & 0.1848 \end{bmatrix} \quad \Omega = \begin{bmatrix} 0.7546 & 0.3979 \\ 0 & 0.7951 \end{bmatrix} \tag{21}$$

$$Z = \begin{bmatrix} 0.8202 & -0.5719 \\ 0.5719 & 0.8203 \end{bmatrix} \quad Q = \begin{bmatrix} 0.6523 & 0.7580 \\ -0.7580 & 0.6523 \end{bmatrix} \tag{22}$$

so the eigenvalues are $\{0.7546/1.0822; 0.7952/0.1848\} = \{0.6972; 4.3028\}$ and apparently the ordering of the system is such that the unstable root 4.3028 is in the lower right corner of Λ and Ω. The other components are

$$\tilde{A} = \begin{bmatrix} 0.2966 & 0.5745 \\ 0.2068 & 0.4006 \end{bmatrix} \quad \tilde{B} = \begin{bmatrix} 0.4944 \\ 0.3447 \end{bmatrix} \tag{23}$$

$$\tilde{C} = \begin{bmatrix} 148.3243 & 0.7580 & 0.1205 \\ 103.4153 & 0.5285 & 1.3031 \end{bmatrix} \tag{24}$$

$$\tilde{W} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad R = \begin{bmatrix} \pounds \\ 1 \end{bmatrix} \quad \tilde{F} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{25}$$

$$x_0 = \begin{bmatrix} 1500 \\ 1500 \end{bmatrix} \qquad \dot{x}_0 = \begin{bmatrix} 1600 \\ 0 \end{bmatrix} \qquad u_1 = \begin{bmatrix} \pounds \\ 0 \end{bmatrix} \tag{26}$$

Note that we have set $E_0 x_1^0 = 1500$. For this simple model we can compute the steady state of the system being $x_1 = 1589.08$ and $u_1 = 17.81$, see Amman and Kendrick (1998). Hence, $U^0 = \{17.81, \ldots, 17.81\}$ is an educated guess for the instruments. The solution to the model is in Table 1.

4. Summary

In this paper we have presented a single agent Linear-Quadratic optimization model that allows for rational expectations. Based on Sims's paper we have used a generalized eigenvalue method for solving the variables that involve the unstable roots. By using an iterative scheme, the reduced model can be fitted into a standard Linear-Quadratic framework that allows us to derive the optimal policy instruments for the model with rational expectations.

References

[1] Amman, H.M. (1996) Numerical Optimization Methods for Dynamic Optimization Problems. In H.M. Amman, D.A. Kendrick & J. Rust (eds.), *Handbook of Computational Economics*, pp. 579-618. Amsterdam: North-Holland Publishers.

[2] Amman, H.M. & D.A. Kendrick (1998) Computing the steady state of the linear quadratic optimization model with rational expectations, *Economics Letters* 58, 185-191.

[3] Amman, H.M., D.A. Kendrick & S. Achath (1995) Solving stochastic optimization models with learning and rational expectations, *Economics Letters* 48, 9-13.

[4] Anderson, G. & G. Moore (1985) A linear algebraic procedure for solving linear perfect foresight models, *Economics Letters* 17, 247-252.

[5] Blanchard, O.J. & C.M. Kahn (1980) The solution of linear difference models under rational expectations, *Econometrica* 48, 1305-1311.

[6] Coleman, T.F. & C. van Loan (1988) *Handbook for matrix computations*. Philadelphia: SIAM.

[7] Fair, R.C. & J. Taylor (1993) Solution and maximum likelihood estimation of dynamic rational expectations models, *Econometrica* 52, 1169-1185.

[8] Fisher, P.G., S. Holly & A.J. Hughes Hallett (1986) Efficient solution techniques for dynamic non-linear rational expectations models, *Journal of Economic Dynamics and Control* 10, 139-145.

[9] Hall, S.G. & S.G.B. Henry (1988) *Macroeconomic modelling*. Amsterdam: North-Holland.

[10] Kendrick, D.A. (1981) *Stochastic Control for Economic Models*. New York: McGraw-Hill.

[11] Kydland, F.E. & E.C. Prescott (1977) Rules are than discretion: the time inconsistency of optimal plans, *Journal of Political Economy* 85, 473-491.

[12] Lucas, R.E. (1976) Econometric policy evaluation: A critique. In K. Brunner and A.H. Meltzer (eds.), The Phillips curve and the labor markets, 19-46. Supplementary series to the *Journal of Monetary Economics*.

[13] Moler, C.B. (1994) Jordan's canonical form just doesn't compute, *The MathWorks Newsletter* 8, 8-9.

[14] Moler, C.B. & G.W. Stewart (1973) An algorithm for for generalized matrix eigenvalue problems, *SIAM Journal on Numerical Analysis* 10, 241-256.

[15] Sims, C. A (1996) Solving linear rational expectations models, *Research paper*. New Haven: Department of Economics, Yale University.

[16] Stewart, G.W. (1973) *Introduction to matrix computations*. New York: Academic Press.

[17] Theil, H. (1968) *Optimal decision rules for government and industry*. Amsterdam: North-Holland.

Table 1.

Solution of the LQ optimization model with RE

t	0	1	2	3	4	5	6	7	8	9	10
x_t	1500	1556	1576	1584	1587	1588	1589	1589	1587	1584	1578
u_t	40	26	21	19	18	18	18	17	16	11	