

**Caution in Macroeconomic Policy:
Uncertainty and the Relative Intensity of Policy**

by

P. Ruben Mercado^{a,*} and David A. Kendrick^b

^a Department of Economics, Bryn Mawr College, Pennsylvania

^b Department of Economics, University of Texas, Austin

Abstract

Two lines of literature show that an increase in the uncertainty will result in a decrease in the vigor of the control variable in the first time period. The first line uses static models and the second dynamic models. In this paper the results in the dynamic line are extended from one-state and one-control models to models with a pair of control variables. We confirm the result of Johansen from the static line that in this case one control will be used less intensely and the other more intensely when current uncertainty is increased. Then we extend this result to models where there are zero weights on the controls to obtain a linear complementarity outcome.

The analysis from both lines of literature concerns essentially single period results since even the studies in the dynamic line have focused on the effects of current uncertainty. In this paper we follow a suggestion from Craine to extend the results to a multiperiod framework. We study the effects of an increase in uncertainty in a future time period on the use of the controls in the first time period by using the Riccati equations. We find that the outcomes, contrary to the single period results, reveal that the control variables (both or at least one, depending of the relative magnitude of first-period control parameter weighted variances) will be used more rather than less intensely.

Keywords: Macroeconomic Policy; Policy Uncertainty; Stochastic Control

JEL classification: C61; E61

*Corresponding author. E-mail: pmercado@brynmawr.edu. Address: Department of Economics, Bryn Mawr College, 101 N. Merion Avenue, Bryn Mawr, Pennsylvania 19010-2899, USA

1. Introduction¹

Recently, there has been a return of interest in the effects of parameter uncertainty on macroeconomic policy, viz. Ghosh and Mason (1994) and Amman and Kendrick (1997).² Strongly related to parameter uncertainty, the issue of “caution” has been studied on both theoretical and empirical grounds.

It seems a reasonable proposition that as the uncertainty of a policy action increases that it would be wise to make less use of that policy. This proposition has been studied by economists who have analyzed how an increase in the uncertainty of a parameter which multiplies a policy variable affects the use of the policy. Beginning from the early paper by Brainard (1969) this work has developed along two lines. The first line used static models of the form:

$$x = Bu \tag{1.1}$$

where:

x = the state vector

u = the control vector

B = an $(n \times m)$ parameter matrix

and the second line used dynamic models with system equations of the form:

$$x_{k+1} = Ax_k + Bu_k \tag{1.2}$$

where:

A = an $(n \times n)$ parameter matrix.

The changes in uncertainty are parameterized as increases in the variance of the B matrix.³ Also this is done in an optimal policy framework with a quadratic criterion function to minimize J in the static case:

¹ We would like to thank three referees for helpful comments on an earlier version of this paper.

² Ghosh and Mason (1994), using empirical multi-country models, study the effects of parameter and model uncertainty within the context of international macroeconomic policy coordination. Amman and Kendrick (1997) have used a small model of the U.S. economy and with numerical rather than analytical methods have found that policy procedures which take account of the uncertainty in parameter estimates perform better on average than those which ignore the uncertainty.

³ See Pohjola (1981) and Don (1983) for a discussion of the appropriate measures of “aggregate” uncertainty in a matrix context.

$$J = x'Wx + u'\Lambda u \quad (1.3)$$

where:

$W =$ an $(n \times n)$ matrix

$\Lambda =$ an $(m \times m)$ matrix

and in the dynamic case with the same terms summed over the time periods. Many of the results in the literature are for the case where $\Lambda = 0$, but some (including some in this paper) are for the case where this matrix is not zero.

The first line of work includes papers by Johansen (1973) and (1978), Young (1975), Pohjola (1981), and Don (1983). Our interest lies in the second line which builds on the work in optimal control theory by Aoki (1967) and Wonham (1969) and which includes papers by Chow (1973), Turnovsky (1975), Shupp (1976a) and (1976b), and Craine (1979). In this dynamic line most of the results are confined to models with a single state and a single control variable. In this paper we move forward the results for the dynamic line to models with a single state and two control variables. Here we confirm in a dynamic setting results obtained by Johansen (1973) and (1978) in the static case for the two-control variable case and extend Johansen's results with a complementary linearity relationship between the two control variables.

Since most of the results in the dynamic line are confined to a single time period, i.e. the last period, the next-to-last period or the first period, it would seem that the results from the two lines of research are really covering the same ground. We decided to use the dynamic approach in hopes that we could extend the results to cover some interaction between time periods. Following a suggestion by Craine (1979), we studied the effects of increases in uncertainty in a future time period on the use of the two control variables in the first time period.⁴ The results differ nicely from the static or single period results which show that an increase in the uncertainty of the parameter associated with a control variable will result in a *decreased* use of that variable in the first time period. In contrast, our results using the Riccati equations in multiperiod models show that if there

⁴ It is standard procedure to focus on the first-period behavior of the control variables. The qualitative behavior of the controls may well change beyond the first period because their optimal values after the first period will be computed recursively, using the model equation to determine the value of the state variable for period " $k+1$ " (see Kendrick 1981, Chapter 6).

is an increase in the uncertainty of a parameter associated with a future control there will be an *increased* use of both, or at least one, of the first-period controls, depending on the relative magnitude of their first-period variances. This occurs because an increase in future uncertainty means that there is relatively less uncertainty associated with the first-period controls.

2. The Problem

We will focus on an optimal policy problem defined as a Quadratic Linear Problem. The state parameter and the control parameters are assumed to be uncertain with no covariance among them. The model parameters and the weights on the state and control variables can be time varying or time invariant.

In formal terms, the problem is expressed as one of finding the controls $(u_k)_{k=0}^{N-1}$ to minimize a quadratic criterion function J of the form:

$$J = E \left\{ \frac{1}{2} x_N' w x_N + \frac{1}{2} \sum_{k=0}^{N-1} (x_k' w x_k + u_k' \Lambda u_k) \right\} \quad (2.1)$$

subject to:

$$x_{k+1} = A x_k + B u_k + \varepsilon_k \quad (2.2)$$

where:

- E = expectation operator
- x = scalar state variable
- u = control variables
- w = positive scalar weight on the state variable
- Λ = diagonal matrix of positive weights on the control variables
- ε = random disturbance

and:

$$A = \begin{bmatrix} A_{11} & 0 \\ 0 & A_{22} \end{bmatrix}$$

$$A = a = \text{scalar}$$

$$B = b = [b_1 \ b_2]$$

$$u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

Desired paths for the state and the controls are zero.⁵ All means and parameter variances are assumed to be known, and a , b , w and Λ can be either fixed or time varying (we have omitted the corresponding time subscripts to simplify the exposition). The solution to the problem is the feedback rule (see Appendix A):

$$u_k = G_k x_k \quad (2.3)$$

where:

$$G_k = \begin{bmatrix} G_{1k} \\ G_{2k} \end{bmatrix} = -\frac{1}{\det} \begin{bmatrix} ab_1 (\Lambda_{22} k_{k+1} + k_{k+1}^2 \sigma_{b_2}^2) \\ ab_2 (\Lambda_{11} k_{k+1} + k_{k+1}^2 \sigma_{b_1}^2) \end{bmatrix} \quad (2.4)$$

and:

$$\begin{aligned} \det = & \Lambda_{11} \Lambda_{22} + \Lambda_{11} k_{k+1} b_2^2 + \Lambda_{22} k_{k+1} b_1^2 + \Lambda_{11} k_{k+1} \sigma_{b_2}^2 + \Lambda_{22} k_{k+1} \sigma_{b_1}^2 \\ & + k_{k+1}^2 b_1^2 \sigma_{b_2}^2 + k_{k+1}^2 b_2^2 \sigma_{b_1}^2 + k_{k+1}^2 \sigma_{b_1}^2 \sigma_{b_2}^2 \end{aligned} \quad (2.5)$$

Consider now the effects on the intensity of the first-period response in the policy variables u when there is a change in the uncertainty (current or future) of one of the parameters which are multiplied by the control variables. We will analyze the case of parameter b_1 and thus $\sigma_{b_1}^2$. Equivalent results can be derived for the case of $\sigma_{b_2}^2$.

3. Current Uncertainty

Let's begin by analyzing the case of an increase in current uncertainty. First look at the effect on u_1 . Thus, from equation (2.4) we obtain:⁶

$$G_1 = \frac{-ab_1 (\Lambda_{22} k_{(1)} + k_{(1)}^2 \sigma_{b_2}^2)}{\det} \quad (3.1)$$

⁵ This is the usual case for log-linearized models or models with variables expressed in percentage changes with respect to a base case.

⁶ To simplify notation, from now on time subscripts corresponding to periods other than zero (that is, the first period) will be between parentheses. Thus, variables and parameters without subscripts between parentheses will correspond to time zero, that is, the first period.

We are interested in the change in $|G_I|$ (the absolute value of G_I) when $\sigma_{b_i}^2$ changes.

Since k is always positive (see Appendix B), given that the weights on the controls were assumed to be positive, and given that the absolute value of quadratic terms is their own value, from (3.1) we can write:

$$|G_I| = \left| \frac{-ab_1(A_{22}k_{(1)} + k_{(1)}^2\sigma_{b_2}^2)}{\det} \right| = \frac{|ab_1|(A_{22}k_{(1)} + k_{(1)}^2\sigma_{b_2}^2)}{\det} \quad (3.2)$$

The elasticity of $|G_I|$ w.r.t $\sigma_{b_i}^2$ is given by:

$$\frac{\frac{\partial |G_I|}{\partial \sigma_{b_i}^2}}{\frac{|G_I|}{\sigma_{b_i}^2}} = - \frac{(k_{(1)}A_{22} + k_{(1)}^2b_2^2 + k_{(1)}^2\sigma_{b_2}^2)\sigma_{b_i}^2}{\det} \quad (3.3)$$

which will always be smaller than zero. Thus, an increase in $\sigma_{b_i}^2$ will always cause a decrease in $|G_I|$, and thus in u_1 , for the first period of the optimization horizon. This means that *the control variable u_1 will always be used less intensely when there is an increase in $\sigma_{b_i}^2$, the current uncertainty associated with “its own” control parameter.*

This result also holds in the opposite direction.⁷

Next consider the case of u_2 . From equation (2.4) we obtain:

$$G_2 = \frac{-ab_2(A_{11}k_{(1)} + k_{(1)}^2\sigma_{b_i}^2)}{\det} \quad (3.4)$$

We are interested in the change in $|G_2|$ (the absolute value of G_2) when $\sigma_{b_i}^2$ changes. As above, we can write:

⁷ This is a well known result for static models and for dynamic models with one state and one control (see for example Chow (1973) and Shupp (1976a) and (1976b)).

$$|G_2| = \left| \frac{-ab_2(\Lambda_{11}k_{(1)} + k_{(1)}^2\sigma_{b_1}^2)}{\det} \right| = \frac{|ab_2|(\Lambda_{11}k_{(1)} + k_{(1)}^2\sigma_{b_1}^2)}{\det} \quad (3.5)$$

The elasticity of $|G_2|$ w.r.t $\sigma_{b_1}^2$ is given by:

$$\frac{\frac{\partial |G_2|}{\partial \sigma_{b_1}^2}}{\frac{|G_2|}{\sigma_{b_1}^2}} = \frac{(\Lambda_{22}k_{(1)}b_1^2 + k_{(1)}^2b_1^2\sigma_{b_2}^2)k_{(1)}\sigma_{b_1}^2}{\det(k_{(1)}\Lambda_{11} + k_{(1)}^2\sigma_{b_1}^2)} \quad (3.6)$$

which will always be positive. Thus, an increase in $\sigma_{b_1}^2$ will always cause an increase in $|G_2|$, and thus in u_2 , for the first period of the optimization horizon. This means that *the control variable u_2 will always be used more intensely when there is an increase in $\sigma_{b_1}^2$, the current uncertainty associated with “the other” control parameter.* This result also holds in the opposite direction.⁸

When there are no weights on the control variables, (3.3) and (3.6) collapse to:

$$\frac{\frac{\partial |G_1|}{\partial \sigma_{b_1}^2}}{\frac{|G_1|}{\sigma_{b_1}^2}} = -\frac{(b_2^2 + \sigma_{b_2}^2)\sigma_{b_1}^2}{(b_1^2\sigma_{b_2}^2 + b_2^2\sigma_{b_1}^2 + \sigma_{b_1}^2\sigma_{b_2}^2)} \quad (3.7)$$

and:

$$\frac{\frac{\partial |G_2|}{\partial \sigma_{b_1}^2}}{\frac{|G_2|}{\sigma_{b_1}^2}} = \frac{b_1^2\sigma_{b_2}^2}{(b_1^2\sigma_{b_2}^2 + b_2^2\sigma_{b_1}^2 + \sigma_{b_1}^2\sigma_{b_2}^2)} \quad (3.8)$$

respectively. Thus, we can predict the percentage change in u_1 and u_2 for the first period of the optimization horizon without the need of any information other than the first-period means of the control parameters and their corresponding variances. Moreover, notice that the elasticity of $|G_2|$ (equation (3.6)) minus the elasticity of $|G_1|$ (equation

⁸ Working with a one-state two-control static model and with no weights on the control variables, Johansen (1973) and (1978 Section 8.10) found a result similar to this.

(3.3)) add up to one. This implies that *first-period percentage changes in u_1 will have a complementary linearity relationship with first-period percentage changes in u_2 .*

A numerical example will help to clarify these findings. Suppose that model's parameter values are given by:

$$a = 0.8, b_1 = 0.9, b_2 = -0.8, \sigma_a^2 = 0.004, \sigma_{b_1}^2 = 1.1, \sigma_{b_2}^2 = 1.155, w = 1$$

and also assume the weights on the controls are zero and the initial value for the state variable x is equal to -400. Assume that there is a 1% increase in the uncertainty associated to u_1 , so that $\sigma_{b_1}^2$ increases to 1.111. Solving the optimization problem for a time horizon of 10 periods, we obtain:⁹

Table 1

<i>time</i> <i>period</i>	u_{1k} ($\sigma_{b_1}^2 = 1.1$)	u_{1k} ($\sigma_{b_1}^2 = 1.111$)	u_{2k} ($\sigma_{b_1}^2 = 1.1$)	u_{2k} ($\sigma_{b_1}^2 = 1.111$)
0	114.307	113.537	-96.768	-97.077
1	39.924	40.052	-33.798	-33.906
2	13.944	13.989	-11.805	-11.843
3	4.870	4.886	-4.123	-4.136
4	1.701	1.707	1.440	-1.445
5-8	0	0	0	0

For the first period we have:

$$\frac{\Delta u_1}{u_1} = \frac{113.537 - 114.307}{114.307} 100 \approx -0.68\%$$

and:

$$\frac{\Delta u_2}{u_2} = \frac{-97.077 - (-96.768)}{-96.768} 100 \approx 0.32\%$$

⁹ The problem was solved with DUALI. See Amman and Kendrick (1996).

i.e. the percent changes are in opposite directions and the sum of their absolute values is one.¹⁰

4. Future Uncertainty

The results presented above correspond to the case of an increase in current uncertainty. They are results which in fact correspond to those that could have been obtained from a static problem. We will now focus on the first-period effects of an increase in future uncertainty, thus dealing with essentially dynamic behavior. As before, we will analyze the case of an increase in the uncertainty associated with parameter b_1 , and thus $\sigma_{b_1(T)}^2$, where T can take any value between 1 and $(N-1)$. Equivalent results can be derived for the case of $\sigma_{b_2(T)}^2$.

First notice that from equations (3.2) and (3.5) we can obtain:

$$\frac{\partial |G_i|}{\partial \sigma_{b_1(T)}^2} = \frac{\partial |G_i|}{\partial k_{(1)}} \cdot \frac{\partial k_{(1)}}{\partial k_{(2)}} \dots \frac{\partial k_{(T-1)}}{\partial k_{(T)}} \cdot \frac{\partial k_{(T)}}{\partial \sigma_{b_1(T)}^2} \quad (4.1)$$

where $i = 1, 2$. From equation (B.3) in Appendix B, we obtain:¹¹

$$\frac{\partial k_{(T)}}{\partial \sigma_{b_1(T)}^2} > 0 \quad \text{and} \quad \frac{\partial k_{(1)}}{\partial k_{(2)}} > 0, \dots, \frac{\partial k_{(T-1)}}{\partial k_{(T)}} > 0 \quad (4.2)$$

and from (3.2) and (3.5) we obtain, respectively:

$$\frac{\partial |G_i|}{\partial k_{(1)}} = \frac{|a b_1| \left\{ \Lambda_{11} \Lambda_{22}^2 + 2 \Lambda_{11} \Lambda_{22} k_{(1)} \sigma_{b_2}^2 + \Lambda_{11} k_{(1)}^2 \sigma_{b_2}^4 + \left[k_{(1)}^2 b_2^2 \left(\Lambda_{11} \sigma_{b_2}^2 - \Lambda_{22} \sigma_{b_1}^2 \right) \right] \right\}}{\det^2} \quad (4.3)$$

and:

¹⁰ Notice that, of course, the behavior of the controls changes for subsequent periods. See footnote 4.

¹¹ The computation of these derivatives is quite involved. However, notice that in (B.3) the denominator of the expression between parenthesis (that is, \det) contains all the same four terms as the ones in the

$$\frac{\partial |G_2|}{\partial k_{(1)}} = \frac{|a b_2| \left\{ \Lambda_{11}^2 \Lambda_{22} + 2 \Lambda_{11} \Lambda_{22} k_{(1)} \sigma_{b_1}^2 + \Lambda_{22} k_{(1)}^2 \sigma_{b_1}^4 + \left[k_{(1)}^2 b_1^2 (\Lambda_{22} \sigma_{b_1}^2 - \Lambda_{11} \sigma_{b_2}^2) \right] \right\}}{\det^2} \quad (4.4)$$

The signs of these derivatives, and thus the signs of (4.1) and those of the corresponding first-period elasticities of $|G_1|$ and $|G_2|$ w.r.t. $\sigma_{b_1(T)}^2$, depend on the sign of $(\Lambda_{11} \sigma_{b_2}^2 - \Lambda_{22} \sigma_{b_1}^2)$ - or, equivalently, on the sign of $(\Lambda_{22} \sigma_{b_1}^2 - \Lambda_{11} \sigma_{b_2}^2)$. Notice that:

$$\text{sign}(\Lambda_{11} \sigma_{b_2}^2 - \Lambda_{22} \sigma_{b_1}^2) = \text{sign} \left[\left(\frac{\sigma_{b_2}^2}{\Lambda_{22}} \right) - \left(\frac{\sigma_{b_1}^2}{\Lambda_{11}} \right) \right] \quad (4.5)$$

Thus, the sign of the elasticities of $|G_1|$ and $|G_2|$ w.r.t. $\sigma_{b_1(T)}^2$, and the corresponding responses of the control variables u_1 and u_2 , will be determined by the relative magnitude of the first-period weighted variances of b_1 (that is, $(\sigma_{b_1}^2 / \Lambda_{11})$) and b_2 (that is $(\sigma_{b_2}^2 / \Lambda_{22})$). There are three possible cases.

If $(\sigma_{b_2}^2 / \Lambda_{22})$ is greater than $(\sigma_{b_1}^2 / \Lambda_{11})$ (opposite-greater weighted variance) then the elasticity of $|G_1|$ will be positive. Thus *in the opposite-greater case an increase in future uncertainty will always induce a more intense first-period response from u_1* . Notice that this result goes in a way against the standard result corresponding to an increase in current uncertainty. We know that when current uncertainty increases, the control variable whose control parameter becomes more uncertain behaves in a more cautious way. On the contrary, when dealing with future uncertainty, we see that the response of that variable becomes more intense.¹² Indeed, it makes sense to take advantage of the relative decrease in first-period uncertainty derived from an increase in future uncertainty, thus making a more intense use of the first-period control since it will now have, in relative terms, a more precise impact on the state variable. However, it makes also sense now to rely more on the first-period control with a more certain (in relative terms within the first period)

numerator. Thus, an increase in $\sigma_{b_1}^2$ or in k_{k+1} will always imply an increase in the denominator greater than the increase in the numerator. Thus, the expression between parenthesis will always increase and so will k_k .

¹² Working with a one-state one-control dynamic model Craine (1979) found a result similar to this.

impact on the state variable. Also notice that the response of $|G_2|$ and thus u_2 will be ambiguous, since the sign of $(\partial|G_2|/\partial k_{(1)})$ will depend, as can be seen in equation (4.6), on the relative magnitude of the first three terms in the numerator sum versus the term between brackets.

If $(\sigma_{b_2}^2/\Lambda_{22})$ is smaller than $(\sigma_{b_1}^2/\Lambda_{11})$ (own-greater weighted variance) then the elasticity of $|G_2|$ will be positive. This means that *in the own-greater case an increase in future uncertainty associated with b_1 (that is, an increase in $\sigma_{b_1(T)}^2$) will always induce a more intense first-period response from “the other” control variable, that is u_2* . As before, it makes sense to have a more intense first-period response to an increase in future uncertainty relying more now on u_2 , the first-period control with a more certain (in relative terms within the first period) impact on the state variable. In this case the response of $|G_1|$ and thus u_1 will be ambiguous since the sign of $(\partial|G_1|/\partial k_{(1)})$ will depend, as can be seen in equation (4.5), on the relative magnitude of the first three terms in the numerator sum versus the term between brackets.

The table below illustrates these findings. Parameter values are the same as in the numerical example in Section 3. Now, $\Lambda_{11} = 1$ and $\Lambda_{22} = 1.05$, and the simulations are carried on for three periods.¹³ The two cases contrast first-period results from baseline simulations against first-period results from experiments in which $\sigma_{b_1(T)}^2$ (the variance of the b_1 parameter in the second period of the simulation) is perturbed with a temporary increase from 1.1 to 11. The results show that in the “opposite-greater” case u_1 is used more intensely, while u_2 is used less intensely. In the “own-greater” case, u_2 is used more intensely, while u_1 is used less intensely.

Table 2

¹³ The experiments were performed with DUALI. See Amman and Kendrick (1996).

	<i>opposite-greater weighted variance</i> $(\sigma_{b_2}^2 = 6)$	<i>own-greater weighted variance</i> $(\sigma_{b_1}^2 = 6)$
<i>u₁ baseline</i>	112.583	29.422
<i>u₁ perturbation</i>	113.883	29.420
<i>u₂ baseline</i>	-24.445	-101.833
<i>u₂ perturbation</i>	-24.412	-102.010

Finally if $(\sigma_{b_2}^2/\Lambda_{22})$ and $(\sigma_{b_1}^2/\Lambda_{11})$ are equal (weighted-variance neutrality) then the elasticities of $|G_1|$ and $|G_2|$ w.r.t. $\sigma_{b_1(T)}^2$ will both be positive. Thus *in the case of weighted-variance neutrality an increase in future uncertainty associated with b_1 (that is, an increase in $\sigma_{b_1(T)}^2$) will always induce a more intense first-period response from both control variables.* It makes sense to take advantage of the relative decrease in first-period uncertainty derived from an increase in future uncertainty, thus making a more intense use of the first-period controls, since they will now have, in relative terms, a more precise impact on the state variable. Since both controls were assumed to have, for the case under analysis, the same first-period weighted parameter variances, it also makes sense to rely on a more intense use of both of them.

The table below illustrates this finding. The experiment is similar to the ones reported in Table 2 above. As expected, both controls are used more intensely in the case of “weighted-variance neutrality”.

Table 3

	<i>Weighted-variance neutrality</i> $(\sigma_{b_1}^2 = 1.1, \sigma_{b_2}^2 = 1.155)$
<i>u₁ baseline</i>	95.235
<i>u₁ perturbation</i>	95.927
<i>u₂ baseline</i>	-80.622
<i>u₂ perturbation</i>	-81.208

5. Extensions

We have employed a dynamic framework in hope that the kind of multiperiod results we have obtained in the article may be extended to more general specifications of the model. In this section we discuss some of the hurdles which must be surmounted for the results to be made more general.

The generalization of the analytical findings presented in this article to models with more than one state and more than two controls may be possible but under specific circumstances. For the case of more than two controls, the determinant “*det*” becomes very involved and, most likely, it will be composed of a sum of positive and negative elements, thus displaying an ambiguous behavior regarding changes in control parameter uncertainty. Allowing for parameter covariances or for non-zero off-diagonal elements in the matrix of weights on the control variables would also imply an ambiguous behavior for “*det*”, unless model parameters and covariances display the appropriate signs. Finally, for the case of more than one state, the first-period response of the controls will be a linear combination between the “feedback-gain coefficients” (the “G’s”) and the corresponding initial conditions for the state variables which may have positive and/or negative signs, thus breaking down the correspondence between changes in the absolute value of the G’s and changes in the control variables. A typical exception to this will be the case in which only one state variable has an initial condition different from zero, which is the usual case when studying optimal policy responses to a unique shock affecting one of the states. For a case like this, the analytical findings of this article will broadly apply, no matter the number of states.^{14, 15}

6. Conclusions

In a literature which has focused on the fact that increases in uncertainty

¹⁴ For the case of more than one state variable, an increase in the uncertainty associated with one of the control variables will mean an increase in the “aggregate” uncertainty of its corresponding vector of parameters. See Pohjola (1981) and Don (1983) for a discussion of the appropriate measures of “aggregate” uncertainty.

¹⁵ For an example of a numerical model with four states and two controls that displays the kind of relative intensity in the use of the controls analyzed in this article, see Mercado and Kendrick (1998). An early numerical example for a two-state two-control variable model was provided by Shupp (1976b).

result in more cautious uses of control variables in a stochastic control framework, we analyze two cases where the response will also include more intense use of some of the control variables.

In both of these results we use models with a single state variable and two control variables. In the first result we confirm in a dynamic model Johansen's result with a static model that an increase in first period uncertainty will result in the more cautious use one of the control variables but a more intense use of the other control variable. We then extend the Johansen results to models where the weights on the control variables are zero to find that the two controls will change in complementary ways so that the absolute value of the changes in the two controls will sum to one.

Even though some of the models used in this field are dynamic almost all of the results are single period, i.e. they focus on the effect of a change in uncertainty on the use of the control variables in the same period. Following a suggestion of Craine and using the Riccati equations, we examine the effect of an increase in uncertainty in a future time period on the use of the control variables in the first time period.

Here we find that the dominant results are not more cautious use of the control variables but rather more intense use. Caution or more intensity in the outcome depends on the ratio of the weight on each control variable to the variance of the parameter which multiplies that control. When these ratios are not equal one of the control variables will always be used more intensely. When these ratios are equal, both control variables will be used more intensely in response to an increase in the future uncertainty associated with one of the control variables.

Appendix A

The solution to the optimization problem is the feedback rule (see Kendrick 1981, Chapter 6):

$$u_k = G_k x_k \quad (\text{A.1})$$

where:

$$G_k = -\left(E\{\Theta_k\}\right)^{-1} \left(E\{\Psi_k\}\right) \quad (\text{A.2})$$

$$E\{\Theta_k\} = \Lambda + E\{b' k_{k+1} b\} \quad (\text{A.3})$$

$$E\{\Psi_k\} = E\{a k_{k+1} b\} \quad (\text{A.4})$$

and where k_{k+1} is the corresponding Riccati matrix which is, in this case, a scalar.

To compute $\left(E\{\Theta_k\}\right)^{-1}$ we proceed as follows. Define:

$$D_k = b' k_{k+1} b \quad (\text{A.5})$$

where D_k is a (2 x 2) matrix, since b is a (1 x 2) vector. We know that (see Kendrick 1981, Chapter 6.5):

$$E\{d_{ijk}\} = k_{k+1} \left[E\{b_i\} E\{b_j\} + \text{cov}(b_i b_j) \right] \quad (\text{A.6})$$

Thus, we have:

$$E\{d_{11k}\} = k_{k+1} (b_1^2 + \sigma_{b_1}^2) \quad (\text{A.7})$$

$$E\{d_{22k}\} = k_{k+1} (b_2^2 + \sigma_{b_2}^2) \quad (\text{A.8})$$

and, since parameter covariances are assumed to be zero:

$$E\{d_{12k}\} = E\{d_{21k}\} = k_{k+1} b_1 b_2 \quad (\text{A.9})$$

Therefore we can write:

$$\left(E\{\Theta_k\}\right)^{-1} = \frac{1}{\det} \begin{bmatrix} \Lambda_{22} + E\{d_{22k}\} & -E\{d_{12k}\} \\ -E\{d_{12k}\} & \Lambda_{11} + E\{d_{11k}\} \end{bmatrix} \quad (\text{A.10})$$

where:

$$det = (\Lambda_{11} + E\{d_{11k}\}) (\Lambda_{22} + E\{d_{22k}\}) - (E\{d_{12k}\})^2 \quad (\text{A.11})$$

is the determinant corresponding to the inversion of $E\{\Theta_k\}$. Substituting equations (A.7), (A.8) and (A.9) into (A.10) and (A.11) we obtain:

$$\left(E\{\Theta_k\}\right)^{-1} = \frac{1}{det} \begin{bmatrix} \Lambda_{22} + k_{k+1}(b_2^2 + \sigma_{b_2}^2) & -k_{k+1}b_1b_2 \\ -k_{k+1}b_1b_2 & \Lambda_{11} + k_{k+1}(b_1^2 + \sigma_{b_1}^2) \end{bmatrix} \quad (\text{A.12})$$

where:

$$det = \Lambda_{11}\Lambda_{22} + \Lambda_{11}k_{k+1}b_2^2 + \Lambda_{22}k_{k+1}b_1^2 + \Lambda_{11}k_{k+1}\sigma_{b_2}^2 + \Lambda_{22}k_{k+1}\sigma_{b_1}^2 + k_{k+1}^2b_1^2\sigma_{b_2}^2 + k_{k+1}^2b_2^2\sigma_{b_1}^2 + k_{k+1}^2\sigma_{b_1}^2\sigma_{b_2}^2 \quad (\text{A.13})$$

Since all parameter means are known, from equations (A.4) and (A.6) we can write:

$$\left(E\{\Psi_k\}\right)' = \left(E\{a \ k_{k+1} \ b\}\right)' = a \ k_{k+1} \ b' \quad (\text{A.14})$$

Finally, substituting equations (A.12) and (A.14) into (A.2) and simplifying, we obtain:

$$G_k = \begin{bmatrix} G_{1k} \\ G_{2k} \end{bmatrix} = -\frac{1}{det} \begin{bmatrix} ab_1(\Lambda_{22}k_{k+1} + k_{k+1}^2\sigma_{b_2}^2) \\ ab_2(\Lambda_{11}k_{k+1} + k_{k+1}^2\sigma_{b_1}^2) \end{bmatrix} \quad (\text{A.15})$$

Appendix B

The Riccati equations are (see Kendrick (1981) pp. 48-49):

$$k_N = w \tag{B.1}$$

for the terminal period and:

$$k_k = w + E\{a^2 k_{k+1}\} - E\{\Psi_k\} \left(E\{\Theta_k\} \right)^{-1} \left(E\{\Psi_k\} \right) \tag{B.2}$$

for any other period. Substituting equations (A.12) and (A.14) into (B.2), simplifying and re-arranging terms, we obtain:

$$k_k = w + a^2 k_{k+1} \left(I - \frac{A_{11} k_{k+1} b_2^2 + A_{22} k_{k+1} b_1^2 + k_{k+1}^2 b_1^2 \sigma_{b_2}^2 + k_{k+1}^2 b_2^2 \sigma_{b_1}^2}{det} \right) \tag{B.3}$$

where det is given by equation (A.13).

The Riccati equations are solved by backward integration starting from period “ $N-1$ ”. Thus, the value of k_{N-1} will be positive, given that $k_N = w$ is positive, that the weights on the controls were assumed to be positive, and that the numerator in the term between parentheses is always smaller than det (which from equation (A.13) is seen to be the case), making the term between parentheses always positive. Given that k_{N-1} is positive, by the same reasoning k_{N-2} will be positive, and so on. Thus, we can conclude that k will always be positive.

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