Information Percolation in Segmented Markets

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Texas Monetary Conference - December 2009
Information Transmission in Markets


- Centralized Exchanges:

- Over-the-Counter Markets:
Contributions of Today’s Paper

1. tractable model of information diffusion in over-the-counter markets with investor segmentation by preferences, initial information, and connectivity.

2. double auction with common values.

3. effects of information and connectivity on profits:
   - more informed/connected investors attain higher expected profits than less informed/connected investors if they can disguise trades.
   - more informed/connected investors may not attain higher expected profits than less informed/connected investors if characteristics are commonly observed.
Outline of the Talk

1. Information Percolation
2. Segmented Markets
3. Double Auction
4. Connectedness and Information
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1. Information Percolation
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Duffie and Manso (2007) and Duffie, Giroux, and Manso (2010):

- Continuum of agents
- Two possible states of nature $Y \in \{0, 1\}$.
- Each agent is initially endowed with signals $S = \{s_1, \ldots, s_n\}$ s.t. $P(s_i = 1 \mid Y = 1) \geq P(s_i = 1 \mid Y = 0)$
- For every pair agents, their initial signals are $Y$-conditionally independent
- Random matching, intensity $\lambda$. 
After observing signals $S = \{s_1, \ldots, s_n\}$, the logarithm of the likelihood ratio between states $Y = 0$ and $Y = 1$ is by Bayes’ rule:

$$
\log \frac{P(Y = 0 \mid s_1, \ldots, s_n)}{P(Y = 1 \mid s_1, \ldots, s_n)} = \log \frac{P(Y = 0)}{P(Y = 1)} + \sum_{i=1}^{n} \log \frac{P(s_i \mid Y = 0)}{P(s_i \mid Y = 1)}.
$$

We say that the “type” $\theta$ associated with this set of signals is

$$
\theta = \sum_{i=1}^{n} \log \frac{P(s_i \mid Y = 0)}{P(s_i \mid Y = 1)}.
$$
What Happens in a Meeting?

- Upon meeting, agents participate in a double auction.

- If bids are strictly increasing in the type associated with the signals agents have collected, then bids reveal type.
Proposition: Let $S = \{s_1, \ldots, s_n\}$ and $R = \{r_1, \ldots, r_m\}$ be independent sets of signals, with associated types $\theta$ and $\phi$. If two agents with types $\theta$ and $\phi$ reveal their types to each other, then both agents achieve the posterior type $\theta + \phi$.

This follows from Bayes’ rule, by which

\[
\log \frac{P(Y = 0 \mid S, R, \theta + \phi)}{P(Y = 1 \mid S, R, \theta + \phi)} = \log \frac{P(Y = 0)}{P(Y = 1)} + \theta + \phi,
\]

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$$= \log \frac{P(Y = 0 \mid \theta + \phi)}{P(Y = 1 \mid \theta + \phi)}$$

By induction, this property holds for all subsequent meetings.
The Boltzmann equation for the cross-sectional distribution $\mu_t$ of types is

$$\frac{d}{dt} \mu_t = -\lambda \mu_t + \lambda \mu_t \ast \mu_t.$$  

with a given initial distribution of types $\mu_0$. 
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with a given initial distribution of types $\mu_0$.

**Proposition:** The unique solution of (10) is the Wild sum

$$\mu_t = \sum_{n \geq 1} e^{-\lambda t} (1 - e^{-\lambda t})^{n-1} \mu_0^n.$$
Proof of Wild Summation

Taking the Fourier transform $\varphi(\cdot, t)$ of $\mu_t$ of the Boltzmann equation

$$\frac{d}{dt} \mu_t = -\lambda \mu_t + \lambda \mu_t \ast \mu_t.$$ 

we obtain the following ODE

$$\frac{d}{dt} \hat{\mu}_t = -\lambda \hat{\mu}_t + \lambda \hat{\mu}_t^2.$$ 

whose solution is

$$\hat{\mu}_t = \frac{\hat{\mu}_0}{e^{\lambda t}(1 - \hat{\mu}_0) + \hat{\mu}_0}.$$ 

This solution can be expanded as

$$\hat{\mu}_t = \sum_{n \geq 1} e^{-\lambda t} (1 - e^{-\lambda t})^{n-1} \hat{\mu}_0^n,$$

which is the Fourier transform of the Wild sum (10).
The Boltzmann equation for the cross-sectional distribution $\mu_t$ of types is

$$\frac{d}{dt} \mu_t = -\lambda \mu_t + \lambda \mu_t^* \mu_t^m.$$

Taking the Fourier transform, we obtain the ODE,

$$\frac{d}{dt} \hat{\mu}_t = -\lambda \hat{\mu}_t + \lambda \hat{\mu}_t^m.$$

whose solution satisfies

$$\hat{\mu}_t^{m-1} = \frac{\hat{\mu}_0^{m-1}}{e^{(m-1)\lambda t} (1 - \hat{\mu}_0^{m-1}) + \hat{\mu}_0^{m-1}}. \quad (1)$$
Groups of 2 (blue) versus Groups of 3 (red)
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Population mass

Current posterior
Groups of 2 (blue) versus Groups of 3 (red)
Groups of 2 (blue) versus Groups of 3 (red)
New Private Information

Suppose that, independently across agents as above, each agent receives, at Poisson mean arrival rate \( \rho \), a new private set of signals whose type outcome \( y \) is distributed according to a probability measure \( \nu \). Then the evolution equation is extended to

\[
\frac{d}{dt} \mu_t = -(\lambda + \rho) \mu_t + \lambda \mu_t \ast \mu_t + \rho \mu_t \ast \nu.
\]

Taking Fourier transforms, we obtain the following ODE

\[
\frac{d}{dt} \hat{\mu}_t = -(\lambda + \rho) \hat{\mu}_t + \lambda \hat{\mu}_t^2 + \rho \hat{\mu}_t \hat{\nu}.
\]

whose solution satisfies

\[
\hat{\mu}_t = \frac{\hat{\mu}_0}{e^{(\lambda+\rho(1-\hat{\nu}))t}(1 - \hat{\mu}_0) + \hat{\mu}_0}.
\]
Other Extensions

► Public information releases
  • Duffie, Malamud, and Manso (2010).

► Endogenous search intensity
  • Duffie, Malamud, and Manso (2009).
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Model Primitives

Same as the previous model except that:

- \( N \) classes of investors.
- Agent of class \( i \) has matching intensity \( \lambda_i \).
- Upon meeting, the probability that a class-\( j \) agent is selected as a counterparty is \( \kappa_{ij} \).
Evolution of Type Distribution

The evolution equation is given by:

\[
\frac{d}{dt} \psi_{it} = -\lambda_i \psi_{it} + \lambda_i \psi_{it} \sum_{j=1}^{N} \kappa_{ij} \psi_{jt}, \quad i \in \{1, \ldots, N\},
\]

Taking Fourier transforms we obtain:

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\frac{d}{dt} \hat{\psi}_{it} = -\lambda_i \hat{\psi}_{it} + \lambda_i \hat{\psi}_{it} \sum_{j=1}^{N} \kappa_{ij} \hat{\psi}_{jt}, \quad i \in \{1, \ldots, N\},
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\]
Special Case: $N = 2$ and $\lambda_1 = \lambda_2$

Proposition: Suppose $N = 2$ and $\lambda_1 = \lambda_2 = \lambda$. Then

\[
\hat{\psi}_1 = \frac{e^{-\lambda t} (\hat{\psi}_{20} - \hat{\psi}_{10})}{\hat{\psi}_{20} e^{-\hat{\psi}_{20}(1 - e^{-\lambda t})} - \hat{\psi}_{10} e^{-\hat{\psi}_{10}(1 - e^{-\lambda t})}} \hat{\psi}_{10} e^{-\hat{\psi}_{10}(1 - e^{-\lambda t})}
\]

\[
\hat{\psi}_2 = \frac{e^{-\lambda t} (\hat{\psi}_{20} - \hat{\psi}_{10})}{\hat{\psi}_{20} e^{-\hat{\psi}_{20}(1 - e^{-\lambda t})} - \hat{\psi}_{10} e^{-\hat{\psi}_{10}(1 - e^{-\lambda t})}} \hat{\psi}_{20} e^{-\hat{\psi}_{20}(1 - e^{-\lambda t})}.
\]
General Case: Wild Sum Representation

**Theorem:** There is a unique solution of the evolution equation, given by

$$\psi_{it} = \sum_{k \in \mathbb{Z}_+^N} a_{it}(k) \psi_{10}^{*k_1} \ast \cdots \ast \psi_{N0}^{*k_N},$$

where $\psi_{i0}^n$ denotes $n$-fold convolution,

$$a_{it}' = -\lambda_i a_{it} + \lambda_i a_{it} \ast \sum_{j=1}^N \kappa_{ij} a_{jt}, \quad a_{i0} = \delta_{e_i},$$

$$(a_{it} \ast a_{jt})(k_1, \ldots, k_N) = \sum_{l=(l_1, \ldots, l_N) \in \mathbb{Z}_+^N, l<k} a_{it}(l) a_{jt}(k-l),$$

and

$$a_{it}(e_i) = e^{-\lambda_i t} a_{i0}(e_i).$$
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Double Auction

- At some time $T$, the economy ends and the utility realized by an agent of class $i$ for each additional unit of the asset is

\[ U_i = v_i Y + v^H (1 - Y), \]

measured in units of consumption, for strictly positive constants $v^H$ and $v_i < v^H$, where $Y$ is a non-degenerate 0-or-1 random variable whose outcome will be revealed at time $T$.

- If $v_i = v_j$, no trade (Milgrom and Stokey (1982)), so that $\kappa_{ij} = 0$.

- Meeting between two agents $v_i > v_j$, then $i$ is buyer and $j$ is seller.

- Upon meeting, participate in a double auction. If the buyer’s bid $\beta$ is higher than the seller’s ask $\sigma$, trade occurs at the price $\sigma$. 

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The prices \((\sigma, \beta)\) constitute an equilibrium for a seller of class \(i\) and a buyer of class \(j\) provided that, fixing \(\beta\), the offer \(\sigma\) maximizes the seller’s conditional expected gain,

\[
E \left[(\sigma - E(U_i | \mathcal{F}_S \cup \{\beta\}))1_{\{\sigma<\beta\} | \mathcal{F}_S}\right],
\]

and fixing \(\sigma\), the bid \(\beta\) maximizes the buyer’s conditional expected gain

\[
E \left[(E(U_j | \mathcal{F}_B \cup \{\sigma\}) - \sigma)1_{\{\sigma<\beta\} | \mathcal{F}_B}\right].
\]

Counterexample: Reny and Perry (2006)
Equilibrium

The prices \((\sigma, \beta)\) constitute an equilibrium for a seller of class \(i\) and a buyer of class \(j\) provided that, fixing \(\beta\), the offer \(\sigma\) maximizes the seller’s conditional expected gain,

\[
E \left[ (\sigma - E(U_i | F_S \cup \{\beta\}))1_{\{\sigma < \beta\} | F_S} \right],
\]

and fixing \(\sigma\), the bid \(\beta\) maximizes the buyer’s conditional expected gain

\[
E \left[ (E(U_j | F_B \cup \{\sigma\}) - \sigma)1_{\{\sigma < \beta\} | F_B} \right].
\]

Counterexample: Reny and Perry (2006)
Lemma: Suppose that each signal $Z$ satisfies

$$\mathbb{P}(Z = 1 \mid Y = 0) + \mathbb{P}(Z = 1 \mid Y = 1) = 1.$$ 

Then, for each agent class $i$ and time $t$, the type density $\psi_{it}$ satisfies the hazard-rate order condition as well as the property

$$\psi_{it}^H(x) = e^x \psi_{it}^H(-x), \quad \psi_{it}^L(x) = \psi_{it}^H(-x) \quad x \in \mathbb{R}.$$
Lemma: For any $V_0 \in \mathbb{R}$, there exists a unique solution $V_2(\cdot)$ on $[v_i, v^H]$ to the ODE

$$
V_2'(z) = \frac{1}{v_i - v_j} \left( \frac{z - v_i}{v^H - z} \frac{1}{h^H_{it}(V_2(z))} + \frac{1}{h^L_{it}(V_2(z))} \right), \quad V_2(v_i) = V_0.
$$

This solution, also denoted $V_2(V_0, z)$, is monotone increasing in both $z$ and $V_0$. Further, $\lim_{V_0 \to v_H} V_2(V) = +\infty$. The limit $V_2(-\infty, z) = \lim_{V_0 \to -\infty} V_2(V_0, z)$ exists. Moreover, $V_2(-\infty, z)$ is continuously differentiable with respect to $z$. 
Bidding Strategies

Proposition: Suppose that \((S, B)\) is an absolutely continuous equilibrium such that \(S(\theta) \leq v^H\) for all \(\theta \in \mathbb{R}\). Let \(V_0 = B^{-1}(v_i) \geq -\infty\). Then,

\[
B(\phi) = V_2^{-1}(\phi), \quad \phi > V_0,
\]

Further, \(S(-\infty) = \lim_{\theta \to -\infty} S(\theta) = v_i\) and \(S(+\infty) = \lim_{\theta \to -\infty} S(\theta) = v^H\), and for any \(\theta\), we have \(S(\theta) = V_1^{-1}(\theta)\) where

\[
V_1(z) = \log \frac{z - v_i}{v^H - z} - V_2(z), \quad z \in (v_i, v^H).
\]

Any buyer of type \(\phi < V_0\) will not trade, and has a bidding policy \(B\) that is not uniquely determined at types below \(V_0\).
Tail Condition

**Definition:** We say that a probability density $g(\cdot)$ on the real line is of exponential type $\alpha$ at $+\infty$ if, for some constants $c > 0$ and $\gamma > -1$,

$$\lim_{x \to +\infty} \frac{g(x)}{x^\gamma e^{\alpha x}} = c$$

In this case, we write $g(x) \sim \text{Exp}_{+\infty}(c, \gamma, \alpha)$. 
Exponential Tails in Percolation Models

Suppose \( N = 1 \), and let \( \lambda = \lambda_1 \) and \( \psi_t = \psi_{1t} \). The Laplace transform \( \hat{\psi}_t \) of \( \psi_t \) is given by

\[
\hat{\psi}_t(z) = \frac{e^{-\lambda t} \hat{\psi}_0(z)}{1 - (1 - e^{-\lambda t}) \hat{\psi}_0(z)}
\]

and \( \psi_t(x) \sim \text{Exp}_{+\infty}(c_t, 0, -\alpha_t) \) in \( t \), where \( \alpha_t \) is the unique positive number \( z \) solving

\[
\hat{\psi}_0(z) = \frac{1}{1 - e^{-\lambda t}},
\]

and where

\[
c_t = \frac{e^{-\lambda t}}{(1 - e^{-\lambda t})^2 \frac{d}{dz} \hat{\psi}_0(\alpha_t)}.
\]

Furthermore, \( \alpha_t \) is monotone decreasing in \( t \), with \( \lim_{t \to \infty} \alpha_t = 0 \).
Proposition: Suppose that, for all $t$ in $[0, T]$, there are $\alpha_i(t)$, $c_i(t)$, and $\gamma_i(t)$ such that

$$\psi_{it}^H(x) \sim \text{Exp}_+^{\infty}(c_i(t), \gamma_i(t), -\alpha_i(t)).$$

If $\alpha_i(T) < 1$, then there is no equilibrium associated with $V_0 = -\infty$. Moreover, if $v_i - v_j$ is sufficiently large and if $\alpha_i(T) > \alpha^*$, where $\alpha^*$ is the unique positive solution to $\alpha^* = 1 + 1/((\alpha^*)^2 \alpha^*)$ (which is approximately 1.31), then there exists a unique strictly monotone equilibrium associated with $V_0 = -\infty$. This equilibrium is in undominated strategies, and maximizes total welfare among all continuous equilibria.
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Class-\(i\) Agent Utility

The expected future profit at time \(t\) of a class-\(i\) agent is

\[
U_i(t, \Theta_t) = E \left[ \sum_{\tau_k > t} \sum_j \kappa_{ij} \pi_{ij}(\tau_k, \Theta_{\tau_k}) \bigg| \Theta_t \right],
\]

where \(\tau_k\) is this agent’s \(k\)-th auction time and \(\pi_{ij}(t, \theta)\) is the expected profit of a class-\(i\) agent of type \(\theta\) entering an auction at time \(t\) with a class-\(j\) agent.

Agents may be able to disguise the characteristics determining their information at a particular auction. In this case, we denote the expected future profit at time \(t\) of a class-\(i\) agent as \(\hat{U}_i(t, \Theta_t)\).
The Value of Initial Information and Connectivity When Trades Can be Disguised

**Theorem:** Suppose that $v_1 = v_2$. If $\lambda_2 \geq \lambda_1$ and if the initial type densities $\psi_{10}$ and $\psi_{20}$ are distinguished by the fact that the density $p_2$ of the number of signals received by class-2 agents has first-order stochastic dominance over the density $p_1$ of the number of signals by class-1 agents, then

$$\frac{E[\hat{U}_2(t, \Theta_{2t})]}{\lambda_2} \geq \frac{E[\hat{U}_1(t, \Theta_{1t})]}{\lambda_1}, \quad t \in [0, T].$$

The above inequality holds strictly if, in addition, $\lambda_2 > \lambda_1$ or if $p_2$ has strict dominance over $p_1$. 
What if Characteristics are Commonly Observed?

- trade-off between adverse selection and gains from trade.
- more informed/connected investor may achieve lower profits than less informed/connected investor.
- If $v_1 = v_2 = 0.9$, $v_3 = 0$, $v^H = 1.9$,
  \[ \psi_{10}(x) = 12 \frac{e^{3x}}{(1 + e^x)^5}, \]
  and $\psi_{20}(x) = \psi_{10} \ast \psi_{10}$.

Then, \[ E[U_2(t, \Theta_1t)] < E[U_1(t, \Theta_2t)] \]
and \[ E[\hat{U}_1(t, \Theta_1t)] < E[U_1(t, \Theta_2t)]. \]
What if Characteristics are Commonly Observed?
Even If Characteristics are Commonly Observed
Connectivity May be Valuable

Proposition: Suppose that $\kappa_1 = \kappa_2$ and $\lambda_1 < \lambda_2$, and suppose that class-1 and class-2 investors have the same initial information quality, that is, $\psi_{10} = \psi_{20}$, and assume the exponential tail condition $\psi_{it}^H \sim \text{Exp}_{+\infty}(c_{it}, \gamma_{it}, -\alpha_{it})$ for all $i$ and $t$, with $\alpha_{10} > 3$,

$$\alpha_{30} > \frac{\alpha_{10} - 1}{3 - \alpha_{10}},$$

and

$$\frac{\alpha_{1t} + 1}{\alpha_{1t} - 1} > \alpha_{3t}, \quad t \in [0, T].$$

If $\bar{\nu} - \nu_3$ is sufficiently large, then for any time $t$ we have

$$\frac{E[\hat{U}_2(t, \Theta_{2t})]}{\lambda_2} > \frac{E[\hat{U}_2(t, \Theta_{2t})]}{\lambda_2} > \frac{E[\hat{U}_1(t, \Theta_{1t})]}{\lambda_1} > \frac{E[U_1(t, \Theta_{1t})]}{\lambda_1}.$$
Subsidizing Order Flow

- Investors \( i \) and \( j \) with \( v_i = v_j \) meet at time \( t \).
- Enter a swap agreement by which the amount
  \[
  k \left[ (p_j(t) - Y)^2 - (p_i(t) - Y)^2 \right],
  \]
  will be paid by investor \( i \) to investor \( j \) at time \( T \).
- Increase connectivity of class \( i \) investors.
- When would investors want to subsidize order flow?
Concluding Remarks

- tractable model of information diffusion in over-the-counter markets.

- initial information and connectivity may or may not increase profits:
  - more informed/connected investors attain higher profits than less informed connected investors when investors can disguise trades.
  - more informed/connected investors may attain lower profits than less informed connected investors when investors’ characteristics are commonly observed.
Other Applications

- centralized exchanges, decentralized information transmission
- bank runs
- knowledge spillovers
- social learning
- technology diffusion