Minimax Regret Treatment Choice with Covariates or with Limited Validity of Experiments

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- fresh revision, probably contains typos! -

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Abstract

This paper continues the investigation of minimax regret treatment choice initiated by Manski (2004). Consider a decision maker who must assign treatment to future subjects after observing outcomes experienced in a finite sample. A certain scoring rule is known to achieve minimax regret in simple versions of this decision problem. I investigate its sensitivity to perturbations of the decision environment in realistic directions. They are: (i) Treatment outcomes may be influenced by a covariate whose effect on outcome distributions is bounded (in one of numerous probability metrics). This is interesting because introduction of a covariate with unrestricted effects leads to a pathological result. (ii) The experiment may have limited validity, for example because of selective noncompliance or because the sampling universe is a potentially selective subset of the treatment population. Thus, even large samples may generate systematically misleading signals. These problems are formalized via a “bounds” approach that turns the problem into one of partial identification.

In both scenarios, small but positive perturbations leave the minimax regret decision rule essentially unchanged. Thus, minimax regret analysis is not knife-edge dependent on ignoring certain aspects of realistic decision problems. Indeed, it recommends to entirely disregard covariates whose effect is believed to be positive but small, as well as small enough amounts of missing data. All findings are finite sample results derived by game theoretic analysis.

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1 Introduction

One recent focus of Charles Manski’s research is to apply statistical decision theory, and in particular the minimax regret criterion, to problems of treatment choice (Manski (2000, 2004, 2006, 2007a, 2007b, 2009); Brock and Manski (2008)). This has spawned a small but active literature\(^1\) with numerous results, not all of which were positive. In particular, Stoye’s (2009a) finding on covariates – to be elaborated below – appear discouraging.

In this paper, I provide two sets of new results that may be considered better news. They seem at first disparate but are driven by the same feature of minimax regret: In comparing the performance of decision rules across possible parameter values, the minimax regret criterion combines a concern for assigning the ex post best treatment, which is only indirectly important under maximin utility, with a concern for the stakes involved in doing so, which is ignored by minimax rules based on “tick” loss functions (e.g., hypothesis tests). In plain English, “getting it right” is intrinsically important, but concern for it depends on the stakes at play.

I reconsider a stylized model of sample-based treatment assignment previously investigated (with slight variations) by Canner (1970) and in a number of recent papers (Hirano and Porter (2009), Manski (2004), Manski and Tetenov (2007), Schlag (2006), Stoye (2009a)). Finite sample minimax regret treatment rules for this problem are known and, by and large, intuitively reasonable. It is also known, however, that an apparently modest modification of the decision problem, namely the introduction of an observable covariate, induces a pathological result (Stoye (2009a)). Specifically, minimax regret then recommends a “no-data rule” that completely ignores the sample at hand. This finding raises at least two questions: Can we redeem minimax regret when there are covariates? More generally, are the minimax regret recommendations previously discovered overly dependent on certain simplifications made in the stylized problem?

I investigate this question by analyzing two different modifications of the problem. First, a covariate is introduced, but its effect on potential outcome distributions is bounded in some distance metric (a menu of such metrics is offered). In a second extension, I relax the assumption that the experiment under consideration has perfect internal as well as external validity. Internal validity is weakened by allowing for selective noncompliance or misclassification. I also take a cue from Manski (2007) and allow sampling populations to differ from treatment populations, albeit in constrained ways. An interesting feature of these modifications is that they turn the decision problem into one of partial identification, that is, the average treatment effect’s sign and thus the identity of the optimal treatment need not be identified.

\(^1\)Other than references cited below, see Brock (2006) and Stoye (2007). Minimax regret was also independently reconsidered by Bergemann and Schlag (2007, 2008), Eozenou et al. (2006), and Chamberlain (2000).
All of these extensions have some interesting features in common. The perturbations of the decision problem can be scaled from zero (i.e. no perturbation) to very large, and a sufficiently large perturbation will induce a pathology in the form of a no-data rule. Yet both cases also lead to very strong “local stability” results: For any sample size \(N\), there exists a positive (albeit of order \(O(N^{-1/2})\)) size of the perturbation such that the finite sample minimax regret treatment rule is essentially or even completely unchanged. Thus, while there are obviously many directions in which the treatment choice problem could be generalized, minimax regret is locally insensitive to generalization in some salient such directions. The result is especially welcome in the case of covariates, where existing findings seem to spell serious trouble. Indeed, covariates with small effect are a situation that minimax regret seems to handle better than some other decision criteria.

All findings are finite sample results derived by analysis of a fictitious game between the decision maker and a malicious Nature. Alternative approaches would be to use large deviations inequalities to find finite sample bounds on regret (Manski (2004)) or to analyze local asymptotic experiments (Hirano and Porter (2009)). Bounds based on the former approach can be very slack; see the web appendix of Stoye (2009a) for numerical illustrations. Asymptotic approximation is doubtlessly an important alternative, not least because finite sample analysis is hard – the present paper contains some rather intricate findings but there are obvious extensions where finite sample analysis eluded this author.

The remainder of this paper is structured as follows. Section 2 sets up the decision problem and recapitulates a quite general result that amalgamates previous work. It is the sensitivity of this result that will be examined. Section 3 contains such analysis with respect to covariates, whereas section 4 considers limitations of the validity of experiments. Section 5 concludes. While intuitions for most results are given in the text, all technical arguments are collected in an appendix.

2 Setting the Stage

2.1 The Decision Problem

The decision problem is as in Manski (2004), and I use the same notation as in Stoye (2009a). A decision maker must assign one of two treatments \(T \in \{0,1\}\) to members \(j\) of a treatment population \(J\). Each member of the treatment population has a response function \(y_j(t) : \{0,1\} \to \{0,1\}\) that maps treatments onto outcomes. Substantively, what this really assumes is that a priori bounds on treatment outcomes exist, are known, and coincide across treatments. Restricting them to lie in \([0,1]\) is then a normalization. The additional restriction to binary outcomes is with less loss of generality than might appear because all treatment rules analyzed in this paper can be extended to general outcomes by a
“binary randomization” technique due to Schlag (2006). Specifically, one can define treatment rules for non-binary outcomes by first replacing every observed outcome with one realization of a binary, mean-preserving spread of itself and then operating rules defined for binary outcomes. This leaves intact minimax regret values and can, therefore, be used to find minimax regret efficiency bounds as well as decision rules that attain them. Perhaps the most important limitation of the setup is that the notation implies the “Stable Unit Treatment Value Assumption” (SUTVA), i.e. outcomes experiences by an individual may depend on treatment received by that individual but not on treatment received by others, excluding externalities or general equilibrium effects.

The population is a probability space \((J, \Sigma_J, P)\) and is “large” in the sense that \(J\) is uncountable and \(P(j) = 0\) for all \(j\). The decision maker cannot distinguish between members of \(J\), hence from her point of view, assigning treatment \(t\) induces a random variable \(Y_t\) (the potential outcome) with distribution \(P(y^j(t))\). We will focus on the distribution \(P(Y_0, Y_1)\) as the unknown quantity. Specifically, a state of the world \(s\) will be identified with \(P(Y_0, Y_1)\), which is partially characterized by a couplet \((\mu_0, \mu_1) \equiv \mathbb{E}(Y_0, Y_1)\). The set \(S\) collects feasible states of the world. Notation will be extended to accommodate covariates as needed.

Assume that if \(s\) were known, the decision maker would resolve the decision problem by maximizing expected outcomes, thus she would assign all subjects to \(T = 1\) if \(\mu_1 > \mu_0\), to \(T = 0\) if \(\mu_1 < \mu_0\), and she would be indifferent if \(\mu_0 = \mu_1\). While this can capture risk aversion (\(Y_t\) might be a utility), it does presumes a utilitarian social welfare function. The decision maker observes not \(s\) but a signal of \(s\), namely treatment outcomes experienced by a random sample of \(N\) members of the treatment population. I take \(N\) to be known, although the generalization to \(N\) being a random variable with known distribution is conceptually simple. The experiment generates a sample space \(\Omega \equiv (\{0, 1\} \times \{0, 1\})^N\) with typical element \(\omega = (t_n, y_n)_{n=1}^N\). Conditional on a realization \(t_n, y_n\) is an independent realization of \(Y_t\). The sampling distribution of \(T\), in turn, depends on the sample design.

I will analyze two experiments that amount to a mild extension of the experiments considered in Tetenove (2009a). Core interest will be in the binomial experiment. Here, \(S = \Delta\{0, 1\}^2\), the set of distributions over \(\{0, 1\}^2\), and sample points are realizations of Bernoulli distributions with parameters \(\mu_0\) or \(\mu_1\), depending on which treatment was assigned. The following designs will be considered:

- **Matched pairs:** Both of \((\mu_0, \mu_1)\) are unknown, \(N\) is even, and \(\{t_1, t_2, \ldots\} = \{0, 1, 0, 1, \ldots\}\).

- **Independent Randomization:** Both of \((\mu_0, \mu_1)\) are unknown, \(N\) is odd or even, and \(\{t_1, t_2, \ldots\}\) are i.i.d. realizations of Bernoulli variables with parameter 1/2 (i.e., independent tosses of fair

\(^2\) More precisely, the technique was first applied to statistical treatment choice by Schlag (2006); it had been independently discovered for related problems by Cucconi (1968), Gupta and Hande (1992), and Schlag (2003). See Stoye (2009a) for an elaboration in this paper’s notation.
coins).

- **Constrained Randomization:** Both of \((\mu_0, \mu_1)\) are unknown, \(N\) is odd or even, and \(\{t_1, t_2, \ldots\}\) is equally likely to be \(\{0, 1, 0, 1, \ldots\}\) or \(\{1, 0, 1, 0, \ldots\}\).

- **Free Treatment Assignment:** Both of \((\mu_0, \mu_1)\) are unknown, \(N\) is odd or even, and within-sample treatment assignment is a choice variable. The decision maker can choose the distribution of \(\{t_n\}_{n=1}^N\) from the set \(\Delta\{0, 1\}^N\) of distributions over \(\{0, 1\}^N\).

Recall that due to the binomial randomization technique, results obtained for the binomial experiment really apply to bounded outcomes more generally, at least in the sense of delineating non-parametric minimax regret efficiency bounds. Nonetheless, I will also report results for a normal experiment. In this experiment, a state of the world is fully described by a vector \((\mu_0, \mu_1)\), the sample space is \(\Omega = \mathbb{R}\), and the decision maker observes a single signal distributed according to \(N(\mu_1 - \mu_0, \sigma)\), where \(\sigma\) is known. Intuitively, the signal is an estimator of the welfare contrast \(\mu_1 - \mu_0\). Results reported for the normal experiment are finite sample in the sense that they precisely apply to this experiment, but the motivation for this experiment lies in asymptotic approximation. For example, consider a sequence of binomial experiments where \(N \to \infty\) and where the true state of the world is of form \((\mu_0, \mu_1) = (\mu + h_0/\sqrt{N}, \mu + h_1/\sqrt{N})\) for some \(\mu \in (0, 1)\) and \((h_0, h_1) \in \mathbb{R}^2\). For any of the above sample designs, elements of this sequence are increasingly well approximated by the normal experiment with parameter values \(\mu_1 - \mu_0 = h_1 - h_0\) and \(\sigma^2 = 4\mu(1 - \mu)/N\). A case of particular interest is the one where \(h_1 = -h_0\) and \(\mu = 1/2\); this turns out to describe sequences of states that are least favorable under the binomial experiment. Thus, minimax regret treatment rules and regret values under the normal experiment with \(\sigma^2 = 1/N\) should be expected to well approximate the corresponding objects for the binomial experiment, an expectation that will be borne out. Beyond this, the normal experiment is of interest because Hirano and Porter (2009) show that it is appropriate for the analysis of a great many experiments, most of which will not be amenable to exact finite sample analysis. One example is the case of the binomial experiment but with “testing an innovation,” that is with known \(\mu_0\), as analyzed in Manski (2004), Stoye (2009a), and Tetenov (2009b), among others. Direct extension of this paper’s analysis to that case appears elusive (and not for lack of trying on this author’s part), but the normal experiment is the appropriate limit experiment for it, so all results reported below apply asymptotically to it.

The decision maker can specify a statistical treatment rule \(\delta : \Omega \mapsto [0, 1]\) that maps possible sample realizations \(\omega\) onto treatment assignments \(\delta(\omega) \in [0, 1]\), where the value of \(\delta\) is interpreted as probability of assigning treatment 1. Nonrandomized decision rules take values only in \(\{0, 1\}\), but

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3 This excludes sequential sample designs, i.e. within-sample treatment assignment cannot depend on lagged (in terms of \(n\)) realizations.
randomization is allowed and will be used. The set of all such decision rules will be denoted by \( D \).

In the case of free treatment assignment, the decision maker can additionally specify an assignment scheme in \( \Delta\{0,1\}^N \) before applying \( \delta \) to the resultant sample; this extension will be suppressed in notation for simplicity. Finally, \( \delta \) is called a no-data rule if it is constant on \( \Omega \) (in words, if no sample information is used).

The expected outcome generated by \( \delta \) given \( \sigma \) is

\[
u(\delta, s) = \mu_0 (1 - \mathbb{E}\delta(\omega)) + \mu_1 \mathbb{E}\delta(\omega),\]

where expectations are taken with respect to the sampling distribution of \( \omega \). Seen as a function of \( s \), \( u(\delta, s) \) is (the negative of) the risk function of treatment rule \( \delta \). Absent a prior on \( s \), attempts to optimize \( u(\delta, s) \) induce a decision problem under ambiguity (Manski (2000)). The two most prominent resolutions of this problem are the Bayesian approach, i.e. to rank decision rules according to \( \int u(\delta, s) d\pi \), where \( \pi \) is a prior on \( S \) (see Chamberlain (2009) for an elucidation and Dehejia (2005) for an application to treatment choice) and the maximin utility approach, i.e. to rank decision rules according to \( \min_{s \in S} u(\delta, s) \). Manski (2004) initiated reconsideration of minimax regret in this context.

To understand this criterion, first define the regret incurred by decision rule \( \delta \) in state \( s \),

\[
R(\delta, s) = \max_{d \in D} u(d, s) - u(\delta, s),
\]

the difference between the expected outcome induced by \( \delta \) and the outcome that could have been achieved, had \( s \) been known. Minimax regret is a maximin ranking with respect to regret loss, thus it recommends to pick

\[
\delta^* \in \arg\min_{\delta \in D} \max_{s \in S} R(\delta, s)
\]

if such a \( \delta^* \) exists, as is the case in examples below. Minimax regret was originally introduced by Savages's (1951) reading of Wald (1950); see Stoye (2009c) for further references on history, axiomatizations, and applications. In order to avoid redundancy, I will jump directly to the core result that is this paper's starting point.

2.2 Existing Results

Exact solutions to the above treatment choice problems are available for both maximin utility and minimax regret. Indeed, one motivation for investigating minimax regret is a pathology of maximin utility: Every decision rule achieves maximin utility because \( \min_{s \in S} u(\delta, s) = 0 \) for all \( \delta \), generated if \( \mu_0 = \mu_1 = 0 \), a degeneracy problem that was diagnosed already by Savage (1954) and, for the present problem or close variations of it, by Hirano and Porter (2009), Manski (2004), Schlag (2006), and Stoye
(2009a).\(^4\) Minimax regret turns out to avoid the problem.

For the binomial experiment, define

\[
\delta^B(\omega) = \begin{cases} 
0, & I_N < 0 \\
1/2, & I_N = 0 \\
1, & I_N > 0
\end{cases}
\]

where

\[
I_N \equiv N_1(\overline{y}_1 - 1/2) - N_0(\overline{y}_0 - 1/2)
\]

\[
\propto [\# \text{ (observed successes of treatment } 1) + \# \text{ (observed failures of treatment } 0)] - [\# \text{ (observed successes of treatment } 0) + \# \text{ (observed failures of treatment } 1)]
\]

with \(N_t\) the number of sample subjects assigned to treatment \(t\), \(\overline{y}_t\) a sample mean that conditions on \(T = t\), and the convention that \(N_t(\overline{y}_t - 1/2) = 0\) if \(N_t = 0\). Thus, treatment is assigned according to the sign of the score \(I_N\), with even randomization if \(I_N = 0\). To get an intuition for this, note that 
\[
\mathbb{E}I_N/N = \Delta \equiv \mu_1 - \mu_0,
\]
that is, \(I_N/N\) is an unbiased estimator of the average treatment effect and its sign therefore a reasonable estimator of the better treatment’s identity. Indeed, if sample design is by either matched pairs or constrained randomization, then \(\delta^B\) simplifies to

\[
\delta^B(\omega) = \begin{cases} 
0, & \overline{y}_1 < \overline{y}_0 \\
1/2, & \overline{y}_1 = \overline{y}_0 \\
1, & \overline{y}_1 > \overline{y}_0
\end{cases}
\]

Finally, for the normal experiment, \(\delta^G = 1\{\omega > 0\}\), where \(\omega \in \mathbb{R}\) is the signal. Recall that \(\omega\) should be intuited as \(\overline{y}_1 - \overline{y}_0\), thus \(\delta^G\) is really very similar to \(\delta^B\); in particular, all are equivalent or similar to the empirical success rules investigated by Manski (2004).

The following result is an amalgam of Canner (1970), Hirano and Porter (2009), Schlag (2006), Stoye (2009a), and Tetenov (2009a).\(^5\)

**Proposition 1** (i) In the binomial experiment, let sample design be any of matched pairs, constrained randomization, or independent randomization. Then \(\delta^B\) achieves minimax regret.

\(^4\)In the abstract, the same types of examples can be constructed for minimax regret (Parmigiani (1992)). They do not seem to occur naturally in models of treatment choice; although, see the section on covariates below.

\(^5\)Part (i) is found in Canner (1970, for matched pairs), Stoye (2009a, for matched pairs and independent randomization), and is also a corollary of results in Schlag (2006, for matched pairs and constrained randomization). Part (ii) minimally expands on Schlag (2006), who recommends constrained randomization. Part (iii) follows from results in both Hirano and Porter (2009) and Tetenov (2009a). Part (i) of the corollary was independently established in Stoye (2009a) and Schlag (2006), a generalization of part (ii) is found in Tetenov (2009a).
In the binomial experiment, let sample design be a choice variable. Then minimax regret is achieved by any of matched pairs, constrained randomization, and independent randomization in conjunction with $\delta^B$.

In the normal experiment, minimax regret is achieved by applying $\delta^G$.

**Corollary 2**

(i) The binomial decision problem has value

$$R^B(N) = \max_{a \in [1/2,1]} \left\{ (2a - 1) \sum_{n<N/2} \binom{N'}{n} a^n (1-a)^{N'-n} \right\}$$

$$N' = \max_{M \in \mathbb{N}} \{ M \leq N : M \text{ is odd} \},$$

where $R^B(0) = 1/2$.

(ii) The Gaussian decision problem has value

$$R^G(\sigma) = \max_{\Delta \in [0,1]} \{ \Delta \Phi(-2\Delta \sigma) \},$$

where $\Phi$ is the standard normal c.d.f.

An important technique to establish this kind of finite sample result is game theoretic analysis (Wald (1945)). Consider a fictitious zero-sum game in which the decision maker picks a decision rule $\delta \in \mathcal{D}$ (possibly at random; note, though, that the result of any randomization over decision rules can itself be expressed as element of $\mathcal{D}$) and Nature picks a state of the world $s$ (possibly at random, meaning that her strategy space is the set $\Delta \mathcal{S}$ of priors $\pi$ over $S$). After both players moved, $s$ is drawn according to $\pi$, $\omega$ is drawn from $s$ according to the relevant sampling scheme, and $\delta$ is operated on $\omega$.

The decision maker then pays $\max_{\delta \in \mathcal{D}} u(d,s) - u(\delta,s)$ to Nature. Any Nash equilibrium $(\delta^*, \pi^*)$ of this game characterizes a minimax regret treatment rule $\delta^*$, and Nature’s equilibrium strategy $\pi^*$ can be interpreted as the least favorable prior that is implicitly selected by the minimax regret criterion.

The least favorable priors underlying proposition 1 are as follows. It turns out that $R(\delta,s)$ depends on $s$ only through $(\mu_0, \mu_1)$, thus identify states with couplets $(\mu_0, \mu_1)$. Then the least favorable prior randomizes evenly over two symmetric states $(a, 1-a)$ and $(1-a, a)$, where $a = \arg \max_{a \in [1/2,1]} \left\{ (2a - 1) \sum_{n<N/2} \binom{N'}{n} a^n (1-a)^{N'-n} \right\}$ in parts (i) and (ii) and $a = \arg \max$ in part (iii). Computation of $a$ is not required for the proofs but leads to the value functions provided in corollary 1.

3 Introducing Covariates

3.1 A Problem with Minimax Regret

Most real-world economic decision and prediction problems involve covariates. The according modification of the benchmark problem is, therefore, of obvious interest. The crucial part of this modification
is that a covariate $X \in \mathcal{X}$ exists and is observable for both sample and treatment subjects, thus the
decision maker can attempt to (implicitly) estimate the mean regression $(\mu_{0x}, \mu_{1x}) = \mathbb{E}(Y_0, Y_1 | X = x)$
and can provide accordingly conditioned treatment recommendations.

To formally introduce a covariate $X$, endow its support $\mathcal{X}$ with a sigma-algebra $\Sigma_X$ and let the
distribution $P$ henceforth be defined on $\Sigma_Y \times \Sigma_X$. In particular, presume that $X$ has a c.d.f. and that
conditional distributions $P(Y_0, Y_1 | X = x)$ are defined for all $x \in \mathcal{X}$. To simplify notation, henceforth
use $(Y_{0x}, Y_{1x})$ to denote random variables with just those conditional distributions. I will initially
assume that $X$ is finite valued, thus $\mathcal{X}$ has $K \geq 2$ elements, but later allow for $X$ to be continuously
distributed on $\mathbb{R}^k$. In either case, the marginal distribution of $X$ is assumed known. Potential outcomes
are, then, random variables $Y_{tx}$ that depend on treatment as well as covariate. A state of the world $s$
is, then, a $\Sigma_X$-measurable function that takes covariate values $x$ into distributions $s_x \equiv P(Y_{0x}, Y_{1x})$;
if $X$ is finite, this simplifies to states $s$ being vectors $(P(Y_{0x}, Y_{1x}))_{x \in \mathcal{X}}$ of length $k$.

Complete specification of the problem also requires a state space. For the case of finite $X$, let
this initially be the perhaps most natural, and largest one, namely $\mathcal{S} = \Delta \{0, 1\}^{2k}$. Define also
$\mathcal{S}_x = \{s_x : s \in \mathcal{S}\}$. A sample $\omega$ collects realizations $(t_n, x_n, Y_n)$, where the distributions of both $T$ and
$X$ depend on the sample design and $Y_n$ is an independent realization of $Y_{t_n, x_n}$. A statistical treatment
rule maps samples $\omega$ into vectors of treatment assignment probabilities $\delta(\omega) \in [0, 1]^k$ whose components
$\delta_x(\omega)$ are identified with probabilities of assigning treatment 1 to subjects with covariate value $x$. A
treatment rule’s risk function is
$u(\delta, s) \equiv \sum_{x \in \mathcal{X}} \mathbb{P}(X = x) (\mu_{0x} (1 - \mathbb{E}\delta_x(\omega)) + \mu_{1x} \mathbb{E}\delta_x(\omega))$, where
$(\mu_{0x}, \mu_{1x}) = \mathbb{E}(Y_{0x}, Y_{1x})$. Regret is $R(\delta, s) \equiv \max_{d \in \mathcal{D}} u(d, s) - u(\delta, s)$ as before.

An obvious question is whether in this extended problem, minimax regret is achieved by pooling
observations across covariates, by conditioning on covariates, or by something in between. The intuitive
trade-off is between the resolution of a decision rule and its sensitivity to sampling variation. Using
large deviations bounds, Manski (2004) discovered that a lower bound on regret incurred by pooling
exceeds an upper bound incurred by conditioning on covariates for rather small sample sizes. The
tentative conclusion was that prevailing practice may err in the direction of too much pooling.

Stoye (2009a) re-analyzed the issue in terms of finite sample regret and found that Manski’s re-
result merely approximates a much stronger, and pathological, one.\footnote{A precise statement requires additional notation not otherwise used in this paper, so I will try to get away with a paraphrase.} Minimax regret recommends to
condition treatment choice on \textit{all} available covariates, even if this leads to empty sample cells. This
conclusion extends to cases of many sample cells and small samples, where the resulting decision rule
is essentially a no-data rule. Indeed, if a covariate takes infinitely many values, then a no-data rule
achieves minimax regret. This result reverses the thrust of previous findings, raising more questions
about minimax regret than about prevailing practice. It motivates this section’s analysis.

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\begin{itemize}
\item \footnote{A precise statement requires additional notation not otherwise used in this paper, so I will try to get away with a paraphrase.}
\end{itemize}
Here is an intuition for why the problem obtains. Both Manski (2004) and Stoye (2009a) use the maximally permissive state space $S = \Delta[0,1]^{2k}$. This state space accommodates priors under which $s_x$ and $s_{x'}$ are independent random variables, and Nature will choose just such priors in the fictitious game. Observations of treatment outcomes for covariate $x$ are then uninformative about potential outcomes for covariate $x'$, thus treatment rules which are Bayes against said priors separate inference across covariates. But the additive separability by covariate of $R(\delta, s)$ can be used to show that priors of this sort, in turn, best respond to these decision rules. An alternative, non-game theoretic intuition is as follows: Ceteris paribus, minimax regret selects for parameter values that make it hard to learn. In the presence of covariates, cross-covariate learning is hardest if cross-covariate signals are vacuous. Indeed, if $X$ is continuous, then uniform learning about the better treatment is impossible, intuitively because the mean regressions $(\mu_{0x}, \mu_{1x})$ can be arbitrarily “wiggly” in $x$, and any minimax-type rule that values learning is expected to encounter trouble. For example, maximin utility or minimax loss with respect to the tick loss function will have the exact same problem.

However, the reason that these problems obtain is that some perhaps highly implausible states (and, by implication, priors) are being contemplated. The least favorable prior driving the above result places equal weight on distributions that render $(\mu_{0x}, \mu_{1x})$ and $(\mu_{0x'}, \mu_{1x'})$ equal and distributions that render them symmetrically opposed. Depending on the substantive meaning of $x$, the latter can be highly implausible. Consider a medical trial in which the race of sample subjects or, even worse, their date of birth is recorded. Clearly, there exists prior information to the effect that these covariates will matter only to a limited degree. In more technical terms, the least favorable prior is demanding of the state space, which must essentially be a Cartesian product of covariate-wise state spaces. The problem may, therefore, lie with (technically) a too permissive state space that (substantively) reflects underspecification of prior information. One might hope that it can be alleviated by introducing plausible prior constraints. The next subsection introduces some such constraints and shows that they have dramatic effects.

### 3.2 Limiting the Effect of Covariates

This section allows for both finite and continuous $X$; in particular, the below result could be rephrased in terms of nonparametric minimax regret mean regression. The innovation is to restrict the effect that $X$ can have on the marginal distribution $P(Y_{lx})$. This will be done through bounding $||P(Y_{lx}), P(Y_{lx'})||$, where $||\cdot||$ can stand for one of several metrics. I begin by analyzing the effect of the following assumption:

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7 More formally, one must have that for any $(s_x)_{x \in X} \times s \in X S_x$, there exists a state $s \in S$ with marginals $(s_x)_{x \in X}$. 
Assumption 1

\[ |\mu_{tx} - \mu_{tx'}| \leq \kappa \]

for all \( x, x' \in X \), \( t \in \{0, 1\} \).

Findings will first be stated for this assumption and then be extended to a number of other probability metrics. Whichever metric is used, some restrictions of this type will be exceedingly plausible in many cases. Again, medical trials constitute a nice example. Researchers would probably be willing to bound the effect of race on outcomes and to quite severely bound the effect of birthdays. At the same time, it will rarely be honest to bound a covariate’s effect at precisely zero. Even birthdays might have a very small effect on reaction to medication (because they proxy for season of birth), so that restricting their effect to be exactly zero is likely an approximation. This is why this section’s main result may be of some interest. In words, it states that for every \( \mathcal{E} \), there exists \( \eta \) that should be thought of as “small but positive” s.t. covariates whose cumulative effect can be bounded by \( \eta \) should be completely ignored. A precise statement is somewhat more long-winded and can be differentiated depending on how much of the sample design is exogenous.

Proposition 3 (i) Consider the binomial experiment. Let \( \tilde{\delta}^B \) be the decision rule that mimics \( \delta^R \) and ignores the existence of \( X \), thus \( \tilde{\delta}^B_{t,x}(\omega) = \delta^B_{t}(\omega) \) for all \( (x, \omega) \) and \( t = 0, 1 \). If the sample design is any of matched pairs, constrained randomization, or independent randomization, the sample is a simple random sample with respect to \( \mathcal{E} \), and assumption 1 holds with \( \kappa \leq 2R^B(N) \), then \( \tilde{\delta}^B \) achieves minimax regret.

(ii) In part (i), if sample design is a choice variable, then independent randomization in conjunction with \( \tilde{\delta}^B \) achieves minimax regret. If sample stratification with respect to \( X \) is a choice variable, simple random sampling with respect to \( X \) achieves minimax regret. If both are choice variables, then simple random sampling with respect to \( X \) in conjunction with independent randomization and use of \( \tilde{\delta}^B \) achieves minimax regret.

(iii) Consider the Gaussian decision problem. Let \( \tilde{\delta}^G \) be just as \( \tilde{\delta}^B \) except that it mimics \( \delta^G \). Then all statements in (i) and (ii) hold true for \( \tilde{\delta}^G \) if \( \kappa \leq 2R^G(\sigma) \).

Again, the proposition essentially states that if the potential effect of \( X \) is small enough, then its presence should be completely ignored.

While the proof involves some tedious algebra, there is an instructive intuition to most of it. To highlight it, I will informally prove a weaker but straightforward version of the result and then indicate how it can be improved. The first big idea is to observe that states can be usefully grouped into two sets: “aligned” states, in which the optimal treatment is the same for all \( x \), and “misaligned” states, in which it is not.
Consider now the problem of maximizing $R(\bar{\sigma}^B, s)$, i.e. Nature’s response problem in the fictitious game. First, the value of this problem must exceed $R^B(N)$ because by choosing states in which $P(Y_{0x}, Y_{1x})$ is constant across $x$, Nature can mimic the least favorable prior from proposition 1 (to which $\bar{\sigma}^B$, in turn, best responds). But it is also easy to see that if $s$ is aligned, then $R(\bar{\sigma}^B, s)$ depends on $s$ only through the marginal success rates $\int \mu_{0x} dP(X)$ and $\int \mu_{1x} dP(X)$, thus the restriction to states where $P(Y_{0x}, Y_{1x})$ is constant is w.l.o.g. among aligned states, and the maximal value of $R(\bar{\sigma}^B, s)$ among aligned states actually equals $R^B(N)$. In short, if Nature has a best response that is aligned, then the fictitious game has an equilibrium that mimics the one from proposition 1 by virtue of both players ignoring the existence of $X$.

Now assume that Nature’s best responses include a misaligned state. In any such state, there exist values $x, x'$ s.t. $\mu_{1x} > \mu_{0x}$ yet $\mu_{1x'} < \mu_{0x'}$. The assumption that $|\mu_{1x} - \mu_{1x'}| \leq \kappa$ then implies that

$$\mu_{1x} - \mu_{0x} \leq \mu_{1x} - \mu_{0x} + \mu_{0x'} - \mu_{1x'} = \mu_{1x} - \mu_{1x'} + \mu_{0x'} - \mu_{0x} \leq 2\kappa$$

and similarly for $\mu_{0x'} - \mu_{1x'}$. Thus, $\sup_{x \in \mathcal{X}} |\mu_{1x} - \mu_{0x}| \leq 2\kappa$. Recalling that $R(\bar{\sigma}^B, s)$ can be informally written as

$$R(\bar{\sigma}^B, s) = \int |\mu_{1x} - \mu_{0x}| \cdot \Pr(\text{the wrong treatment is assigned for subjects with } X = x) dP(X),$$

it follows that the supremum of $R(\bar{\sigma}^B, s)$ over misaligned states cannot exceed $2\kappa$. If $R^B(N) > 2\kappa$, Nature is, then, better off playing aligned states, leading us again to the equilibrium from proposition 1. This informally establishes proposition 3 for the case of $(\mu_{0x}, \mu_{1x})$ unknown and with the caveat that the conclusion applies only if $\kappa \leq R^B/2$. The only fundamental “missing piece” relative to proposition 3 is that the threshold claimed is $\kappa \leq 2R^B(N)$, corresponding to an improvement of the bound on regret over misaligned states by a factor of 4. This involves tedious algebra, but some intuitions are as follows. First, to achieve $R(\bar{\sigma}^B, s) = 2\kappa$ would require $\int |\mu_{1x} - \mu_{0x}| dP(X) = 2\kappa$ and, therefore, that $|\mu_{1x} - \mu_{0x}| = 2\kappa$ for almost every $x$. Noting that (1) really implies $|\mu_{1x} - \mu_{0x}| + |\mu_{1x'} - \mu_{0x'}| \leq 2\kappa$, this is possible only if $\mu_{1x} - \mu_{0x} = 2\kappa$ almost everywhere on $\mathcal{X}$ and $\mu_{1x} - \mu_{0x} = 0$ otherwise (or vice versa).  

This state is aligned, however – if it were Nature’s best response in (the closure of) the set of misaligned acts, one would be back to proposition 1. Let us therefore restrict attention to “properly misaligned” states in which both $\mu_{1x} > \mu_{0x}$, and $\mu_{1x} < \mu_{0x}$ occur on non-null events in $\mathcal{X}$. Assuming for the moment that there exists an event $E \subset \mathcal{X}$ with $P(X \in E) = 1/2$, a natural conjecture for an optimal such state would be the symmetric one where $(\mu_{0x'}, \mu_{1x}) = (0, \kappa)$ on $E$ and $(\mu_{0x'}, \mu_{1x}) = (\kappa, 0)$ otherwise. This state has an appealing symmetry, fully exploits whatever leeway (1) gives, and ensures

---

8Regret in this state will actually fall well short of $2\kappa$ because $\bar{\sigma}_1$, while not a best response, performs quite well in it; but this observation is not essential.
that $\delta^G$ picks up no signal – the decision maker might literally just as well flip a coin. The proof is wrapped up by verifying the conjecture and thereby a bound of $\kappa/2$.

### 3.3 Additional Results for Special Cases

A complete analysis of this decision problem seems elusive, but some additional results are available for special cases. Most importantly, a complete solution is available for the case where $X$ is continuously distributed and both $(\mu_{0x}, \mu_{1x})$ are unknown. In this case, $\delta^B$ is the minimax regret rule for any $\kappa$.

**Proposition 4** Consider the binomial/Gaussian experiment. Let $X$ be continuously distributed on $\mathbb{R}$. Then $\delta^B, \delta^G$ achieves minimax regret for all $\kappa$. The minimax regret value of the decision problem is $\max\{R^B(N, \kappa/2)\} [\max\{R^G(N, \kappa/2)\}]$.

The core observation underlying this finding is that where $X$ is continuously distributed and both $(\mu_{0x}, \mu_{1x})$ are unknown, Nature can always enforce a regret that is arbitrarily close to $\kappa/2$. The idea is simple: Pick some event $E \in \Sigma_X$ s.t. $\Pr(X \in E) = 1/2$ and let $(\mu_{0x}, \mu_{1x})$ equal $((1 - \kappa)/2, (1 + \kappa), 2)$ on $E$ and $((1 + \kappa)/2, (1 - \kappa), 2)$. It is easy to see that $\delta^*$ and any other decision rule that fails to take $X$ into account will attain regret of $\kappa/2$. Of course, if the decision maker knew $E$, her best response would be otherwise. However, Nature can randomize over states of this type that vary by selection of $E$, and if $X$ is distributed continuously on a sufficiently rich support, then uniform learning about $E$ is elusive. As a result, $\delta^*_{\infty}$ is minimax regret optimal in this case, although admittedly only by virtue of being as bad as many other rules. It is worth mentioning, though, that while the best no-data rule for this case also achieves an infimal regret of $\kappa/2$, $\delta^*$ weakly outperforms this rule in any state and does so strictly in many states, so it establishes inadmissibility of the no-data rule.

An immediate implication of proposition 5 is that whenever an event $E$ as just explained exists, then proposition 3 is “tight” in the sense that if $\kappa > R^B(N)$, then the minimax regret value of the problem exceeds $R^B(N)$. This is because Nature can then respond to $\delta^B$ with a state of the form just explained. Proposition 5 establishes that no-data rules are inadmissible in that proposition’s setting. The following result makes a similar and in some sense more general point.

**Proposition 5** Consider any of the settings of proposition 3(i)-(iii). If assumption 1 holds with some $\kappa < 1$, then any no-data rule incurs strictly more than minimax regret.

One might argue that proposition 5, and part (v) of proposition 3, substantially alleviate the “no-data problem” previously identified may be substantially alleviated. In contrast, the older finding that minimax regret precludes cross-covariate inference for finite $X$ enjoys some robustness. Specifically, let $X$ be discrete and imagine holding $\kappa$ fixed and driving $N$ to infinity. Then it stands to reason...
that as cell sizes expand, cross-covariate inference “fades out” in some way. In particular, one may imagine that in some appropriately conceived limit experiment, the minimax regret rule would not reflect any cross-covariate inference whatsoever. But in fact, a much stronger result is available: The “no cross-covariate inference” finding from Stoye (2009a) is fully reinstated for a finite sample size.

Proposition 6 Fix any \( \kappa > 0 \). Let \( X \) be finite. The following findings apply to both the binomial and Gaussian experiments.

(i) There exists a number \( N^B(\kappa) \) \([N^G(\kappa)]\) s.t. for any sample stratification \((N_1, \ldots, N_K)\) with \( \min_{k=1}^K N_k \geq N^B(\kappa) \) \([N^G(\kappa)]\), the minimax regret rule applies \( \delta^B \) \([\delta^G] \) separately for each value of \( X \).

(ii) Consider random sampling with respect to \( X \). Then there exists (a different) \( N^B(\kappa) \) \([N^G(\kappa)]\) s.t. covariate-wise application of \( \delta^B \) \([\delta^G] \) achieves minimax regret for \( N \geq N^B(\kappa) \) \([N^G(\kappa)]\).

While perhaps a bit cumbersome to state formally, the result is easy to verbalize: For any \( \kappa \), there is a critical sample size (depending on the sample design and in the case of random stratification the distribution of \( X \)) beyond which minimax regret will discourage cross-covariate inference. An intuition for this is as follows. For any value \( x \in X \), the least favorable prior supporting the “no-covariates” result randomizes evenly over \((\mu_{0x}, \mu_{1x}) = ((1 + \Delta_{N_x})/2, (1 - \Delta_{N_x})/2) \) and \(((1 - \Delta_{N_x})/2, (1 + \Delta_{N_x})/2)\), where the worst-case signal strength \( \Delta_{N_x} \) depends on the sample size allocated to \( x \) (or its distribution if \( N_x \) is a random variable). It can be shown that \( \Delta_{N_x} \rightarrow 0 \) as \( N_x \rightarrow \infty \), indeed \( \Delta_{N_x} = O(N_x^{-1/2}) \). This is true for all values \( x \), so if all cell sizes diverge, then the least favorable prior becomes consistent with \( |\mu_{tx} - \mu_{tx'}| \leq \kappa \) for some finite \( N \). (In the case of random sampling with respect to \( X \), \( N_x \) is of course random, but the argument can be adapted.)

As a practical illustration of these findings, consider an example from Stoye (2009a): There is a binary covariate, \( X \in \{m, f\} \) say, the decision maker can freely choose the sample design, and both treatments are unknown. Then if the effect of \( X \) is unrestricted, the decision maker will want to separately sample males and females and to conduct entirely separate inference (according to proposition 1) within either group, no matter how small the overall sample size.

Table 1 illustrates how this conclusion is affected by limiting the effect of \( X \). Columns of the table refer to different overall sample sizes \( N \). The first row, labelled, \( \kappa^B(N) \), indicates the threshold at which proposition 3 begins to apply. In words, if one is willing to assume \( |\mu_{tx} - \mu_{tx'}| \leq \kappa^B(N) \), then \( X \) should be (just) ignored. The second row indicates upper thresholds \( \kappa^B(N) \) beyond which proposition 6 applies, that is, minimax regret inference is completely separate across covariates. Rows three and four display the analogous quantities for the normal experiment with \( \sigma = 1/2 \) and rescaled by \( N^{-1/2} \). This localization around the binomial experiment’s least favorable prior amounts to an asymptotic approximation of the binomial experiment, and the point of rows three and four is to show, by means
of comparison with the first two rows, whether this approximation performs well.

Some interesting observations to be made from table 1 are as follows. First, the benchmark decision rules are rather robust to the existence of covariates with small effects. For \( N = 10 \), if one is willing to bound \( |\mu_{tm} - \mu_{t,f}| \) by 10% — which would appear reasonable in many contexts —, the finite sample minimax regret recommendation is to completely ignore gender. A bound on \( |\mu_{tm} - \mu_{t,f}| \) of 1% — still reasonable for some treatments and certainly if \( X \) were date of birth, say — would justify ignoring the covariate at a rather substantive sample size of \( N = 1000 \). At the same time, once samples grow large, the separation result does “kick in” for moderate \( \kappa \), thus it is not specious. For \( N = 500 \), if one believes an effect of gender of 5% to be plausible, one should (from the minimax regret point of view) completely separate samples. While the intermediate cases are, unfortunately, extremely hard to solve, table 1 illustrates that the minimax regret treatment choice problem is resolved for a rather wide range of parameter values. It is also seen that the normal approximation performs very well, with agreement to three and more significant digits beginning at \( N = 100 \).

### 3.4 Extending the Result to other Distance Metrics

The preceding results can be extended to other notions of distance between probability distributions. Fix any distributions \( P \) and \( Q \) on the Borel sets \( B(\mathbb{R}) \) on the real line. Define the Total Variation distance \( ||·||_{TV} \) by \( ||P, Q||_{TV} = \max_{E \in B(\mathbb{R})} |P(E) - Q(E)| \), the log odds ratio distance by \( ||P, Q||_{LOR} = \max_{E \in B(\mathbb{R})} |\log(P(E)/Q(E))| \), and the Kullback-Leibler divergence (a.k.a. relative entropy) by \( D_{KL}(P||Q) = \int \log(dP/dQ)dp \) (assuming existence of these quantities).\(^9\) Then the following obtains:

\(^9\)If \( P \) and \( Q \) are Bernoulli with parameters \( \mu_P \) and \( \mu_Q \), then these simplify to \( ||P, Q||_{TV} = |\mu_P - \mu_Q| \) (and this then also coincides with the Kolmogorov-Smirnov and other distances), \( ||P, Q||_{LOR} = |\log((\mu_P(1-\mu_Q))/(\mu_Q(1-\mu_P)))| \), and \( D_{KL}(P||Q) = \mu_P \log(\mu_P/\mu_Q)+\mu_Q \log((1-\mu_P)/(1-\mu_Q)) \). The proof of proposition 3 uses these simplifications, but the result does not materially depend on binary outcomes. Also, bounding the log odds ratio can alternatively be justified by imposing a logit model of treatment outcomes and limiting the coefficient on \( X \) in the model as in example 4 later in this paper.
Proposition 7

(i) Assume that \( ||P(Y_{tx}), P(Y_{tx'})||_{TV} \leq \kappa \) for all \( x, x' \in \mathcal{X} \), where \( \kappa > 0 \) is known. Then the conclusion of proposition 3 obtains.

(ii) Assume that \( ||P(Y_{tx}), P(Y_{tx'})||_{LOR} \leq \gamma \) for all \( x, x' \in \mathcal{X} \), where \( \gamma > 0 \) is known. Then the conclusion of proposition 3 obtains if

\[
\gamma \leq 2 \log \frac{1 + 2R_1^*(N)}{1 - 2R_1^*(N)} \quad \text{or} \quad \gamma \leq 2 \log \frac{1 + 4R_2^*(N)}{1 - 4R_2^*(N)},
\]

depending on whether one or both treatments are unknown, and with the caveat that the threshold may not be best possible if \( \mu_0 \) is known.

(iii) Assume that \( D_{KL}(P(Y_{tx})|P(Y_{tx'})) \leq \delta \) for all \( x, x' \in \mathcal{X} \), where \( \rho > 0 \) is known. Then the conclusion of proposition 3 obtains if

\[
\rho \leq 2R_1^*(N) \log \frac{1 + R_1^*(N)}{1 - R_1^*(N)} \quad \text{or} \quad \rho \leq 4R_2^*(N) \log \frac{1 + 2R_2^*(N)}{1 - 2R_2^*(N)},
\]

depending on whether one or both treatments are unknown, and with the caveat that the threshold may not be best possible if \( \mu_0 \) is known.

The proposition can be established by showing that bounds in these different metrics induce bounds on \( |\mu_{tx} - \mu_{tx'}| \) and then invoking proposition 3 (modulo a change in variables for (ii) and (iii)). The proof strategy might seem inefficient because bounds on Kullback-Leibler divergences and log odds ratios are strictly stronger than the implied bounds on differences in means, thus there might be slack in the threshold values given above. But this is not so: The least favorable priors which render proposition 3 tight also maximize \( |\mu_{tx} - \mu_{tx'}| \) subject to the respective bound on \( D_{KL} \) or \( ||| \) or \( LOR \).

Thus, the thresholds given in proposition 4 are tight in the case of two unknown treatments, and in the case of testing an innovation, cannot be improved without using knowledge of a particular \( \mu_0 \). One could, therefore, think of bounding \( |\mu_{tx} - \mu_{tx'}| \) as an efficient approach in the sense of getting the desired effect through the weakest among a set of restrictions. Also, as \( |\mu_{tx} - \mu_{tx'}| = 1 \) implies infinite log odds ratio distance as well as Kullback-Leibler divergence, one can eliminate no-data rules by imposing any finite bound on either quantity.

These results are of interest not least because the existence of a covariate with unrestricted domain causes problems for some other decision criteria. Similarly to previous remarks, the mean regression of interest \( (\mu_{0x}, \mu_{1x}) \) may have a sufficiently rich domain that a noninformative prior appears elusive, but there is a Bayesian interpretation to this section’s results. Consider a robust (multi-prior) Bayesian who is willing to allow any prior s.t. \( |\mu_{tx} - \mu_{tx'}| \leq \kappa \) holds with probability \( 1 \).\(^{10}\) Propositions 7 and 8 applies to this Bayesian’s \( \Gamma \)-minimax regret decision; thus, a robust Bayesian might want to completely ignore the covariate as well.

\(^{10}\)Note this is written as a restriction on parameter values. It is not sufficient to impose a similar condition for prior expectations.
4 Introducing Imperfect Experiments

The benchmark example presumes a perfectly valid experiment: Treatment assignment is externally randomized, and causal inference from treatment to effect is not hindered by selective noncompliance, misclassification of treatment status, and the like. In short, the experiment is internally valid. Furthermore, the sampling universe coincides with the population to which treatments will be futurely assigned, thus causal inference can be extrapolated to the treatment population. In short, the experiment is externally valid. I will now relax both of these assumptions (although separately) by proposing models that allow for some failure of either internal or external validity. While it would be conceptually feasible to do so by writing explicit, fully identified models of selection and then conducting minimax regret analysis, this road will not be taken below. One interesting feature of maximin criteria is that by offering a decision theoretic resolution to ambiguity, they do not force the user to resolve all such ambiguity via identifying assumptions. This feature will here be exploited by models that do not imply statistical identifiability of the average treatment effect or even of its sign and hence, of the better treatment's identity. The analysis thereby connects decision theoretic analysis of treatment choice with another literature pioneered by Manski, namely partial identification.\footnote{See Manski (2003) for a survey. Minimax regret treatment choice has also been linked to partial identification analysis in Brock (2006), Stoye (2007, 2009b), Manski (2007a, 2007b, 2009), and Tetenov (2009). Of these, only Manski (2007a) contains a finite sample result.}

Contrary to the preceding section, I will present full equilibrium analyses that provide exact minimax regret treatment rules for all possible values of underlying parameters. Two interesting features of the findings will be as follows: First, as was the case with covariates, minimax regret treatment rules completely ignore small but positive departures from the benchmark model. Second, for departures that take the decision problem outside of this “robustness region,” minimax regret is achieved by discarding sample data until one is back in the robustness region. The latter feature may appear surprising at first, but has a clear intuition that will be elaborated.

4.1 Limited Internal or External Validity: Some Partially Identified Models

Limitations to internal validity occur when causal inference toward the sampling population is not warranted because data were not obtained in a perfect experiment. Typical causes of this problem are selection into treatment or selective noncompliance with assigned treatment as well as misclassification of treatment status. For the purpose of treatment choice, what is important about all of these problems is the precise wedge they drive between observable treatment outcomes and treatment outcomes that
would be observed in perfect experiments. To illustrate, consider the following examples.\footnote{The appendix contains proof of claims made in the examples.}

**Example 1 Selective Noncompliance as a Partial Identification Problem**

Let treatment assignment be at random, but assume that a probability mass $\varepsilon$ of subjects are non-compliers, that is, they may not receive the assigned treatment. Assume that assigned treatment is observed but received treatment is not. Let $D \in \{0, 1\}$ denote treatment assigned. If $\Delta = \mathbb{E}(Y_1|D = 1) - \mathbb{E}(Y_0|D = 0)$ is the (readily observable) intention-to-treat effect (ITE) and $\tilde{\Delta} = \mathbb{E}(Y_1 - Y_0)$ is the average treatment effect (ATE) of interest, then tight bounds on the latter in terms of the former are

$$
\tilde{\Delta} \in [\Delta - 2\varepsilon, \Delta + 2\varepsilon] \cap [-1, 1].
$$

It is clear from the proof that the same bounds apply if one observes received but not assigned treatment, an example being social experiments in which treatment was assigned at random, treatment received is recorded, but there is a worry that some sample subjects switched treatment groups without this being recorded in the data (e.g., in the Tennessee STAR experiment). They also apply to observational studies if the “treatment” variable is a binary covariate that is assumed exogenous but is observed subject to classification error (e.g., the setting of Hill and Kreider (2009)).

The above bounds are achieved by assuming that all noncompliers are “anti-compliers” whose received treatment never agrees with their assigned treatment, and that the average treatment effect for noncompliers is either $-1$ or $+1$.\footnote{This scenario has observable implications: It implies that both $\mu_1$ and $\mu_0$ lie in $[\varepsilon, 1 - \varepsilon]$, and therefore might be eventually learned to not apply. This does not affect minimax regret analysis. A similar remark applies to example 2 below.}

**Example 2 Selective Monotonic Noncompliance as a Partial Identification Problem**

In the setting of example 1, assume monotonicity as just defined, then worst-case bounds on $\tilde{\Delta}$ improve to

$$
\tilde{\Delta} \in [\Delta - \varepsilon, \Delta + \varepsilon] \cap [-1, 1].
$$

Very roughly speaking, the length of the identified set is cut in half because for each of the two noncomplier subpopulations, one of the two relevant potential outcomes is in fact observed.
Limitations to internal validity are not the only problem with experiments. In his recent monograph, Manski (2007) mentions external validity, i.e. the question whether conclusions that are valid within an experimental population can be extrapolated beyond the lab. In Manski’s words, “an experiment is said to have external validity if the distribution of outcomes realized by a treatment group is the same as the distribution of outcomes that would be realized in an actual program” (p. 26). However, “participation in experiments ordinarily cannot be mandated in democracies. Hence experiments in practice usually draw subjects at random from a pool of persons who volunteer to participate. So one learns about treatment response within the population of volunteers rather than within the population of interest” (p. 138).

To explicitly model such a concern, assume that the experiment is internally valid, yet the sampling population (that is, the universe from which samples are drawn) may be a selective subset of the treatment population (that is, the population whose treatment responses are of ultimate interest). Obviously, interesting results are available only if the former is at least somewhat informative about the latter,\textsuperscript{14} and the relation between the two will be accordingly constrained. Two partially identifying models of selection into sampling are as follows.

**Example 3 Selection into Sampling as a Missing Data Problem**

The sampling population is a subset of the treatment population. No assumption is made about how this subset is selected; however the subset has mass $1 - \varepsilon$, where $\varepsilon$ is known. Thus, if $Z \in \{0, 1\}$ indicates membership in the sampling population, then the distribution $P(Y_0, Y_1)$ of potential outcomes in the treatment population is characterized as

$$P(Y_0, Y_1) = (1 - \varepsilon)P(Y_0, Y_1|Z = 1) + \varepsilon P(Y_0, Y_1|Z = 0).$$

Then tight bounds on $\hat{\Delta}$ are

$$\hat{\Delta} \in [(1 - \varepsilon)\Delta - \varepsilon, (1 - \varepsilon)\Delta + \varepsilon] \cap [-1, 1],$$

where $\Delta$ is again the ITE.

**Example 4 Selection into Sampling as a Hidden Covariate Problem**

Do not constrain the amount of missing data, but let the decision to join the sampling population (e.g., to volunteer) be modelled via a logit model, i.e.

$$\Pr(Z = 1|X = x) = \frac{\exp(x\gamma)}{1 + \exp(x\gamma)}.$$  

\textsuperscript{14}Perhaps this is not as obvious as one might have thought. Woorall (2007) relates the case of benoxaprofen, where a dramatic failure of external validity occurred because the medication was mainly used on elderly patients, but these were not at all sampled into the randomized clinical trials.
where the covariate $X \in [0, 1]$ is unobservable. The effect of $X$ on treatment response is unconstrained, but $\gamma \in [0, c]$, where $c$ is known.

Under this selection model, the ATE $\Delta$ can be tightly bounded by

$$\exp(c) - (1 - \exp(c)) \frac{\mathbb{E}(Y_1|Z = 1)}{\mathbb{E}(Y_1|Z = 1)} - \frac{\mathbb{E}(Y_0|Z = 1)}{\mathbb{E}(Y_0|Z = 1)} \leq \Delta \leq \exp(c) - (1 - \exp(c)) \frac{\mathbb{E}(Y_0|Z = 1)}{\mathbb{E}(Y_0|Z = 1)} \cdot$$

The first of these models formalizes selection into the sampling universe as a missing data problem. If only a small subset of the treatment population select themselves into the sampling population, this will naturally be insufficiently restrictive to allow for interesting conclusions. (In particular, it is easy to see that a no-data rule obtains if $\varepsilon = 1$.) In contrast, the hidden covariate model can allow for such conclusions even if the sampling population is a small subset of the treatment population. Of course, this is because this model is more stringent along other dimensions. While the effect of $X$ on treatment response is completely unconstrained, hence $X \in [0, 1]$ is a vacuous assumption as far as distortion of treatment responses goes, the bound on $\gamma$ combines with the upper limit on $X$ to constrain the effect of $X$ on selection probabilities. Specifically, variation in $X$ can change conditional odds ratios of entering the treatment population by a factor of at most $\exp(c)$.

For all of these models, I will take $\varepsilon$ respectively $c$ to be known to the researcher. If they can only be bounded, then it is w.l.o.g. to set them equal to their respective upper bound, because in both cases, identification decays monotonically as the parameter increases. It is conceivable that while not strictly known, $\varepsilon$ and/or $c$ are identifiable from data other than the ones under consideration. In other contexts, they are probably best thought of as a user-specified “plausibility” or “sensitivity” parameter; such sensitivity parameters are by now quite common in both biostatistics and econometrics.

4.2 Results and an Interpretation

Examples 1-3 are similar in that they induce bounds on $\Delta$ that (up to inclusion in $[-1, 1]$) are linear functions of $\Delta$. Specifically, in all three cases, these bounds can be written as

$$\Delta \in [a\Delta - b, a\Delta + b] \cap [-1, 1]$$

for some $a \in (0, 1]$ and $b > 0$. This section’s first main result provides minimax regret treatment rules for any such model.

**Proposition 8** Assume partial identification of $\Delta$ is as in (2). Then:

(i) Consider the binomial experiment with independent randomization. Let $\delta^R_M$ denote treatment rule $\delta^R$ applied on the first $M$ data points only. Let $N' = \max\{M \leq N : M \text{ odd}\}$, and $\gamma_N \equiv$
$2^{-N'}\sum_{n>N'/2} \binom{N'}{n}(2n-N')$, noting that $\gamma_1 = 1/2$. Minimax regret is achieved by

$$\delta^*(\omega) = \begin{cases} 
\delta^B(\omega) & \text{if } b \leq \frac{a}{2N}, \\
\alpha^*\delta^B_N(\omega) + (1 - \alpha^*)\delta^B_{N'-2}(\omega) & \text{if } \frac{a}{2N} < b < a, \\
\frac{\alpha}{\sigma} \cdot \delta^B_1(\omega) + (1 - \frac{\alpha}{\sigma}) \cdot \frac{1}{2} & \text{if } a \leq b < 1, \\
\frac{1}{2} & \text{if } b \geq 1.
\end{cases}$$

(3)

Here

$$N^* = \min \{ M : \gamma_M > a/2b \},$$

$$\alpha^* = \frac{a/2b - \gamma N^*-2}{\gamma N^*-\gamma N^*-2}.$$ 

(ii) Consider the binomial experiment with free within-sample treatment assignment. Then minimax regret is achieved by independent randomization in conjunction with $\delta^*$ as just defined.

(iii) Consider the normal experiment. Minimax regret is achieved by

$$\delta^*(\omega) = \begin{cases} 
\delta^G(\omega) & \text{if } b \leq (2/\pi)^{1/2}\sigma a, \\
\Phi(\omega; 0, (2/\pi)^{1/2}(b/a) - \sigma) & \text{if } (2/\pi)^{1/2}\sigma a < b < 1, \\
\frac{1}{2} & \text{if } b \geq 1.
\end{cases}$$

Corollary 9 (i) The minimax regret value of the binomial decision problem equals

$$R^*(N; a, b) = \begin{cases} 
2^{-N'} \max_{\Delta \in [0,(1-b)/a]} \left\{ (a\Delta + b) \sum_{n \leq (N'-1)/2} \binom{N'}{n}(1 + \Delta)^n(1 - \Delta)^{N'-n} \right\} & \text{if } b \leq \frac{a}{2N}, \\
\frac{1}{2} \min \{ b, 1 \} & \text{otherwise.}
\end{cases}$$

(ii) The minimax regret value of the normal decision problem equals

$$R^*(\sigma; a, b) = \begin{cases} 
\max_{\Delta \in [0,(1-b)/a]} \left\{ (a\Delta + b)\Phi(-\Delta; 0, 2\sigma) \right\} & \text{if } b \leq (2/\pi)^{1/2}\sigma a, \\
\frac{1}{2} \min \{ b, 1 \} & \text{otherwise.}
\end{cases}$$

The proposition may appear algebraically involved, but there is a clear intuition. To see it, inspect (3) and imagine increasing $b$ while $a$ and $N$ are fixed. This corresponds to reducing the (worst-case) information contained in the data. For $b$ small enough, $\delta^B$ applies unchanged, i.e. small enough distortions to external validity should be completely ignored. This robustness result links this section’s analysis to the preceding section one’s, although it comes with the caveat that it applies for independent randomization only; the other sample designs do not in general achieve minimax regret here.

As $b$ increases, the worst-cased wedge – which is of course attained under the least favorable prior – between $\Delta$ and $\bar{\Delta}$ increases, thus the signal generated by sampling from a population characterized by treatment effect $\Delta$ becomes less and less informative about $\bar{\Delta}$. Once $b$ exceeds a certain threshold, this
fact has an interesting consequence: If the decision maker “optimally” followed the signal, she would make herself vulnerable to exploitation by Nature through states of the world in which $\Delta$ and $\bar{\Delta}$ have different signs, thus the signal generated by the sampling population has a misleading tendency. The intuition is clear in an extreme case: Imagine that $a \approx 0$ and $b$ is large, yet sample size is sufficiently large that the sign of $\Delta$ is correctly represented in the sample with probability near 1. Then the maximal regret incurred by using $\delta^B$ is close to $b$, induced by states where $\Delta = 1$ yet $\bar{\Delta} = -b$. Clearly the decision maker wants to avoid this trap. One salient way to do so is to ensure that the signal is not followed too closely. More specifically, for every $(a, b)$, there exists a maximal sample size s.t. $\delta^B$ still achieves minimax regret. Intuitively, the mixture rule in the second line of (3) throws away data until this threshold is reached. The randomization reflects the fact that the threshold “sample size” which just deters Nature from using misleading signals is generically a mixture of two adjacent ones.

The expression in the third line of (3) may at first look different, but is really in the same spirit: It reflects the case where even a sample size of $N = 1$ induces a “too informative” signal. Close inspection reveals that the third case prescribes mixing the benchmark rule, applied to the first sample point only, with the no-data rule that always flips a coin, thus it can be thought of as using an effective sample size of less than 1. An extreme is reached in the last line, where a no-data rule corresponds to an effective sample size of 0.

In terms of the underlying equilibrium and also the decision problem’s value function, the most important change occurs as one moves from the first to the second line of (3). Indeed, the reader may wonder how minimax regret can ever prescribe to discard data, given that the decision rules have to be Bayes against some prior. The answer is that as soon as $\delta^B$ fails to achieve minimax regret, the least favorable prior renders the data completely uninformative, thus in the fictitious game’s equilibrium, the decision maker’s strategy is completely unconstrained, and she could as well use all or no data points. As is typical with mixed strategy equilibria (which these randomizations over different effective sample sizes really are), the decision maker’s equilibrium strategy is really determined by her opponent’s indifference condition.

The theme that using too much information eventually makes the decision maker vulnerable is connected to an independent finding by Tetenov (2009b), who establishes that in certain situations of partial identification, the returns in terms of minimax regret to improving identification completely dominate returns to increasing sampling precision once sampling precision exceeds a certain threshold. The result is driven by a similar mechanism, a possible intuition being that if very much sample information is available, then fully exploiting it makes the planner too predictable in scenarios where said information may be misleading. The same theme also emerges in the interesting detail that proposition 5 generally recommends independent randomization and not matched pairs or constrained randomization. While these do achieve minimax regret for $\varepsilon = 0$ and also for sufficiently small $\varepsilon$, numerical
evaluation reveals that there are parameter values where they fail to do so, specifically if \( N \) is just below \( N_e \). The reason is that under some off-equilibrium priors, sample design by matched pairs is more efficient than independent randomization in estimating \( \Delta \). This added precision is arguably attractive in the benchmark problem, where it implies that constrained randomization weakly dominates independent randomization; in the extended scenario, it can make the decision maker vulnerable to manipulation.

The finite sample minimax regret treatment rule for the hidden covariate model has a similar intuition.

**Proposition 10** Assume partial identification of \( \Delta \) is through the hidden covariate model. Then:

(i) Consider the binomial experiment with independent randomization sample design. Minimax regret is achieved by

\[
\delta^* (\omega) = \begin{cases} 
\delta^B (\omega) & \text{if } c < \log \left( \gamma^{-1} + \sqrt{1 + \gamma^{-2}} \right), \\
\alpha^* \delta_{N^*} + (1 - \alpha^*) \delta_{N^*-2} & \text{if } \log \left( \gamma^{-1} + \sqrt{1 + \gamma^{-2}} \right) \leq c < \log \left( 2 + \sqrt{5} \right), \\
\frac{1}{2} + \frac{2 \exp (c)}{\exp (2c)} & \delta_1^B (\omega) \quad \text{if } c \geq \log \left( 2 + \sqrt{5} \right). 
\end{cases}
\]

(ii) Consider free within-sample treatment assignment. Then minimax regret is achieved by independent randomization in conjunction with the above decision rule.

(iii) Gaussian case: To be added.

**Corollary 11** The binomial experiment’s minimax regret value is

\[
R^* (N; c) = \\
\begin{cases} 
2^{-N'} \max_{\Delta \in [0,1]} \left\{ \frac{(1+\Delta) \exp (c) - (1-\Delta)}{(1+\Delta) \exp (c) + (1-\Delta)} \sum_{n \leq N'-1} \binom{N'}{n} (1+\Delta)^n (1-\Delta)^{N'-n} \right\} & \text{if } c < \log \left( \gamma^{-1} + \sqrt{1 + \gamma^{-2}} \right), \\
\frac{2 \exp (c)}{\exp (2c)+1} & \text{otherwise.}
\end{cases}
\]

Gaussian case: to be added.

I conclude this section with a numerical illustration of the analysis and some remarks about its limitations.

The threshold distortions \( \varepsilon \) respectively \( \exp (c) \) below which \( \delta^B \) applies are displayed in table 2. (The analogous numbers derived from the Gaussian experiment are omitted for brevity.) For example, consider again the missing data scenario from example 3. Then if \( N = 10 \), minimax regret prescribes to ignore missing data of up to 29%, a scenario in which the signal generated by the sample population could be rather misleading. The corresponding number for a sample size of \( N = 100 \) still exceeds 10%.

Some limitations of this section’s analysis are as follows. First, and less importantly, minimax regret decision rules are essentially unique (up to subtleties about tie-breaking) only as long as they
Table 2: Numerical illustration: Threshold values from corollaries 7 and 10.

<table>
<thead>
<tr>
<th>N</th>
<th>1</th>
<th>3</th>
<th>5</th>
<th>10</th>
<th>20</th>
<th>50</th>
<th>100</th>
<th>200</th>
<th>500</th>
<th>1000</th>
</tr>
</thead>
<tbody>
<tr>
<td>ε (ex. 1)</td>
<td>0.5</td>
<td>0.333</td>
<td>0.267</td>
<td>0.203</td>
<td>0.142</td>
<td>0.089</td>
<td>0.063</td>
<td>0.044</td>
<td>0.028</td>
<td>0.020</td>
</tr>
<tr>
<td>ε (ex. 2)</td>
<td>1</td>
<td>0.667</td>
<td>0.533</td>
<td>0.406</td>
<td>0.284</td>
<td>0.178</td>
<td>0.126</td>
<td>0.089</td>
<td>0.056</td>
<td>0.040</td>
</tr>
<tr>
<td>ε (ex. 3)</td>
<td>0.5</td>
<td>0.4</td>
<td>0.348</td>
<td>0.289</td>
<td>0.221</td>
<td>0.151</td>
<td>0.112</td>
<td>0.082</td>
<td>0.053</td>
<td>0.038</td>
</tr>
<tr>
<td>exp (ex. 4)</td>
<td>4.23</td>
<td>3.00</td>
<td>2.53</td>
<td>2.10</td>
<td>1.71</td>
<td>1.42</td>
<td>1.28</td>
<td>1.19</td>
<td>1.12</td>
<td>1.08</td>
</tr>
</tbody>
</table>

coincide with $\delta^B$ or $\delta^G$, i.e. as long as learning occurs. Once a noninformative equilibrium has been reached, one encounters a plethora of minimax regret treatment rules. For an obvious example, the decision maker could use any appropriately sized subset of sample points rather than the first ones. Earlier versions of this paper contained yet other rules which are available from the author.15

Second, closed-form analysis is feasible because the bounds on $\Delta$ are highly tractable. In contrast, consider the situation where both assigned and received treatment are observable. Bounds on $\Delta$ are then known, but are characterized as the solution to a linear programming problem involving observable features of the joint distribution of assigned treatments, received treatments, and outcomes (Balke and Pearl (2000)). These solutions are well-behaved as functions of identifiable quantities, so it is expected that minimax regret treatment rules based on them will have qualitatively similar features to the above. But a full closed-form analysis appears elusive.

5 Conclusion

Manski (2004) placed minimax regret treatment choice on the agenda of a small but active literature in econometrics. This literature was quite successful in characterizing finite sample minimax regret experimental designs and treatment rules in stylized problems, but recently also contained some findings that would have to worry proponents of minimax regret. In particular, no-data recommendations, which are a core argument against maximin utility in this context, affect minimax regret as soon as covariates are introduced.

This paper extended previous findings in the direction of more realism, substantially extending the range of finite sample analysis. It also provided findings that may be more charitable to minimax regret.

15A notable case occurs in proposition 7 whenever the third case obtains, i.e. whenever $\delta^* = \frac{1}{2} \cdot \delta^P + (1 - \frac{1}{2}) \cdot \frac{1}{2}$. Here $E\delta^*(\omega)$ is linear in $(\mu_1 - \mu_0)$, and any unbiased estimator of $(\mu_1 - \mu_0)$ can substitute for $\delta^P$. A salient example is to rather use $\frac{1 + I_N/N}{2}$, i.e. the sample analog of $(\mu_1 - \mu_0)$ generated from the full sample. This alternative solution specifically applies in the missing data scenario whenever $\varepsilon > 1/2$: It is then the sample analog of a rule discovered by Manski (2007a) for the case of known $\Delta$. Furthermore, Manski (2007a) observed that under conditions very similar to $\varepsilon > 1/2$ here (in particular, the identity of the better treatment is known to be unlearnable), this specific rule is finite sample minimax regret optimal.
While the introduction of covariates with arbitrarily large effects as well as of unrestricted disagreement between response functions in sample and treatment population causes massive problems for minimax regret (and, indeed, for any other decision rule), minimax regret is locally insensitive to these features. To repeat, the interesting finding is not that the effect of these features on minimax regret treatment rules approaches zero as their potential magnitude approaches zero; such a result should be expected from any reasonable treatment rule. It is not obvious, however, that as in the cases analyzed here, said effect becomes exactly zero for some positive (and in some cases, arguably quite large) magnitude of the effect. Furthermore, these findings are finite sample results, so they cannot be artifacts of approximations of any kind. (They are limited, however, by not applying to the case of testing an innovation with imperfect experimental data.)

An interesting feature is the “stop collecting data” result for situations of sufficiently large limitations to data. The result is robust in that similar issues appear in related but different contexts (Tetenov (2009b)), and it may at first raise eyebrows – econometricians are surely not used to discarding data. But on second thought, it may not be so clear that this result is unreasonable. Without fully identifying assumptions, there is a limit – precisely described by partial identification analysis – on how informative the signal generated by the data can be at any sample size. This fact holds true irrespective of which decision criterion one uses. The discovery is merely that under a minimax regret criterion, the limit is fully attained for some finite sample size.

Regarding their methodological significance, I finally emphasize that these results are driven by a specific feature of minimax regret not shared by many other decision criteria. Bayesianism and maximin utility attempt to maximize some increasing function of the risk function $u(\delta, s)$, thus they intrinsically care about “good” versus “bad” states but not about how much efficiency loss – meaning underperformance relative to what could have been achieved ex post – a decision rule causes in a given state. Tick loss functions that have the opposite intuition: They are only sensitive to whether the ex post correct decision was made and not to the extent of utility loss caused. Minimax regret seems unique in combining aspects of both, reflecting a consideration for ex post optimality but also for the stakes at play. The local robustness features of minimax regret decision rules can be directly linked to this feature, that is, proof intuitions rely on it. This link is some cause for optimism that similar robustness could be uncovered with respect to other modifications of the problem. It is hoped that along with axiomatic discussions and empirical applications, findings like these advance our understanding of the trade-offs involved in choosing among treatment rules and experimental designs for real-world, sample based decision problems.
A Proofs

Least Favorable Priors Underlying Proposition 1

Preliminaries. This proposition synthesizes existing results, so a detailed proof will be omitted. However, an explanation that in particular exhibits the least favorable priors is helpful. Recall that results are established by analyzing the following, fictitious game: The decision maker (DM) picks a statistical decision rule \( \delta \in \mathcal{D} \) and Nature picks a state of the world \( s \). Both players may randomize and therefore employ mixed strategies. As \( \mathcal{D} \) is closed under randomization, mixed strategies for DM can be identified with elements of \( \mathcal{D} \). A mixed strategy for Nature is a distribution \( \pi \in \Delta \mathcal{S} \) that can be thought of as a prior. After both players moved, \( s \) is drawn from \( \mathcal{S} \) according to distribution \( \pi \), \( \omega \) is drawn from \( s \) according to the relevant sampling scheme, and \( \delta \) is operated on \( \omega \). Nature’s payoff (and DM’s loss) is

\[
R(\delta, s) = \max_{d \in D} u(d, s) - u(\delta, s).
\]

Any Nash equilibrium \((\delta^*, \pi^*)\) of this game characterizes a minimax regret treatment rule \( \delta^* \) and least favorable prior \( \pi^* \). Note also the following simplification: The minimax regret treatment rule \( \delta^* \) must minimize

\[
\max_{d \in D} u(d, s) - u(\delta, s) = \max_{d \in D} \{ \mu_1 \mathbb{E}d(\omega) + \mu_0 (1 - \mathbb{E}d(\omega)) \} - (\mu_1 \mathbb{E}\delta(\omega) + \mu_0 (1 - \mathbb{E}\delta(\omega))) ,
\]

which is clearly equivalent to maximization of \( \mu_1 \mathbb{E}\delta(\omega) + \mu_0 (1 - \mathbb{E}\delta(\omega)) \), thus \( \delta^* \) must be Bayes given prior \( \pi^* \), a condition that is typically easy to verify. The tricky part in most proofs is to verify Nature’s best-response condition, namely that all states \( s^* \) in the support of \( \pi^* \) maximize \( R(\delta^*, s) \).

It will turn out that regret depends on states of the world \( s \) only through \((\mu_0, \mu_1)\). Anticipating this observation, states will be identified with these parameters, so a pure strategy for Nature is an act \((\mu_0, \mu_1) \in [0, 1]^2\).

(i) Let the sample design be independent randomization. Assume that \( N \) is odd. Then a Nash equilibrium of the fictitious game is given by \((\delta^B, \pi^*)\), where \( \pi^* \) randomizes evenly over \( \{(a^*, 1 - a^*), (1 - a^*, a^*)\} \) for some \( a^* > 1/2 \) to be characterized. To verify this, note that Nature’s best-response problem against \( \delta^B \) is

\[
\max_{(\mu_0, \mu_1) \in [0, 1]^2} \max \left\{ \left( \mu_1 - \mu_0 \right) F \left( \frac{N-\mu_1}{2}; N, \frac{1+\mu_1-\mu_0}{2} \right) , \left( \mu_0 - \mu_1 \right) \left( 1 - F \left( \frac{N-\mu_1}{2}; N, \frac{1+\mu_1-\mu_0}{2} \right) \right) \right\} ,
\]

where \( F(\cdot; N, p) \) denotes the binomial c.d.f. with indicated parameters. The expression uses that \( \delta^B \) can alternatively be written as \( \delta^B(\omega) = 1 \{ z > 1/2 \} \), where \( z \equiv (1 + I_N/N)/2 \) is distributed as a binomial variable with parameters \((N, (1 + \mu_1 - \mu_0)/2)\). The objective is symmetric in \((\mu_0, \mu_1)\),
Thus \((a^*, b^*)\) maximizes it iff \((b^*, a^*)\) does; furthermore, \((\mu_0, \mu_1)\) enters the objective only through 
\(\Delta \equiv \mu_1 - \mu_0\). It follows that one best response to \(\delta^B\) is \(\pi^*\) as indicated, where 
\(a^* = (1 + \Delta^*)/2\) and \(\Delta^*\) solves 
\[
\max_{\Delta \in [0, 1]} \phi(\Delta), \phi(\Delta) = \Delta \Phi\left(\frac{N - 1}{2}; N, \frac{1 + \Delta}{2}\right). \tag{4}
\]
This establishes equilibrium. The value of the game is \(R^B(N) = \phi(\Delta^*)\), which upon algebraic evaluation 
leads to the corollary.

Now let \(N\) be even. \(I_N\) is necessarily even in this case, so if \(I_N \neq 0\), \(\delta^B(\omega)\) is invariant to dropping 
the last observation from the sample. If \(I_N = 0\), the 50/50 tie-breaking can equivalently be achieved by dropping 
the last observation from the sample. Thus if the planner plays \(\delta^B\), Nature’s best-response problem is as if \(\delta^*\) 
were applied to a sample of size \((N - 1)\), and \(\pi^*\) (with \(\Delta^*\) computed for sample size \(N - 1\)) is a best response. 
On the other hand, \(\delta^B\) is a best response to \(\pi^*\) for reasons just explained. Thus an equilibrium has been found, 
and \(R^*(N) = R^*(N - 1)\) in this case.

Consider now constrained randomization, then it is easy to see that \(\delta^*\) continues to be a best 
response to \(\pi^*\). Brute force algebra in Schlag (2006) establishes that \((a^*, 1 - a^*)\) maximizes 
\(E\delta^*(\omega)\) among states of the form \((a, a - \Delta^*)\), hence \(\pi^*\) remains a best response to \(\delta^B\). Matched pairs coincide 
with constrained randomization if \(N\) is even.

\(\text{(ii)}\) Let sample design be a choice variable. Then \(\delta^B\) combined with any sample design is a best 
response to \(\pi^*\), but \(\pi^*\) is a best response to \(\delta^B\) in combination with the above (and possibly other) 
sample designs, thus a Nash equilibrium has been found.

\(\text{(iii)}\) The argument is much the same, except that \(1 - E\delta^B(\omega) = \Phi(\Delta/2, \sigma) = \Phi(-\Delta; 0, 2\sigma)\), 
where \(\Phi(: \mu, \sigma)\) stands for the normal c.d.f. with indicated parameters. Thus, \(\Delta^*\) solves 
\[
\max_{\Delta \in [0, 1]} \Delta \Phi(-\Delta; 0, 2\sigma).
\]

**Corollary 2** Follows by evaluating (4).

**Proposition 3**

\(\text{(i)}\) The claim is that if \(R^B(N) \geq 2\kappa\), then an equilibrium of the fictitious game is given by 
\((\delta^B, \pi^*)\), where \(\pi^*\) is a prior that has \(s_x = s_{x'}\) a.s. for all \(x, x'\) and that otherwise mimics \(\pi^*\). In 
words, the equilibrium mimics the equilibrium from proposition 1, with both players ignoring the 
existence of \(X\). The claim will be established by verifying the equilibrium. This will be done explicitly 
for independent randomization with \(N\) odd; the extension to other cases is as in proposition 1. In 
this game, pure strategies for Nature are given by states of the world \(s\). Strictly speaking, these are
where the potential outcomes \( \mu_{0x}, \mu_{1x} \) depend on \( s \) only through the implied mean regressions \( (\mu_{0x}, \mu_{1x}) = E(\mu_{0x}, \mu_{1x}) \in [0, 1]^2 \) however, so identify states with those. Also define the unconditional expectations \( \mu_i = \int \mu_{ix} dP(X) \).

Nature’s response to \( \delta^B \) must be supported on states that solve

\[
\max_{s \in S} R(\delta^B, s) = \max_{s \in S} \int \left( \max\{\mu_{0x}, \mu_{1x}\} - \mu_{0x}(1 - E_0^B(\omega)) - \mu_{1x} E_{1x}^B(\omega) \right) dP(X)
\]

\[
= \max_{s \in S} \int \left( E_0^B(\omega) \cdot (\mu_{0x} - \mu_{1x}) \cdot 1\{\mu_{0x} > \mu_{1x}\} + (1 - E_{1x}^B(\omega)) \cdot (\mu_{1x} - \mu_{0x}) \cdot 1\{\mu_{1x} \geq \mu_{0x}\} \right) dP(X)
\]

\[
= \max_{s \in S} \left\{ E_0^B(\omega) \int (\mu_{0x} - \mu_{1x}) \cdot 1\{\mu_{0x} > \mu_{1x}\} dP(X) + (1 - E_{1x}^B(\omega)) \int (\mu_{1x} - \mu_{0x}) \cdot 1\{\mu_{1x} \geq \mu_{0x}\} dP(X) \right\}
\]

\[
= \max_{s \in S} \left\{ E_0^B(\omega) pa + (1 - E_{1x}^B(\omega))(1 - p)b \right\},
\]

where the first step substitutes in for the definition of \( R(\delta^B, s) \) and the next two steps rearrange terms, using that that \( \delta^B(\omega) \) does not depend on \( x \).

Define

\[
a = \frac{\int (\mu_{0x} - \mu_{1x}) 1\{\mu_{0x} > \mu_{1x}\} dP(X)}{\int 1\{\mu_{0x} > \mu_{1x}\} dP(X)},
\]

\[
b = \frac{\int (\mu_{1x} - \mu_{0x}) 1\{\mu_{0x} \leq \mu_{1x}\} dP(X)}{\int 1\{\mu_{0x} \leq \mu_{1x}\} dP(X)},
\]

\[
p = \int 1\{\mu_{0x} \leq \mu_{1x}\} dP(X).
\]

Recall also that \( \delta^B(\omega) = 1\{z > N/2\} \), where \( z \) is distributed binomially with parameters \( (N, (1 + \Delta)/2) \) and therefore depends on \( s \) only through the ITE \( \Delta = \mu_1 - \mu_0 = (1 - p)b - pa \). For the remainder of this proof only, define \( f(\Delta) = f((1 - p)b - pa) = E\delta^B(\omega) \), then Nature’s best-response problem can be written as

\[
\max_{s \in S} R(\delta^B, s) = \max_{s \in S} \{ pa f((1 - p)b - pa) + (1 - p)b(1 - f((1 - p)b - pa)) \}
\]

This objective depends on \( s \) only through \( (a, b, p) \).

Call a state “aligned” if \( (\mu_{0x} - \mu_{1x})(\mu_{0x'} - \mu_{1x'}) \geq 0 \) for all \( x, x' \in X \), thus the optimal treatment is the same across covariate values. If the solution to (5) is achieved by an aligned state that induces values \( (a^*, b^*, p^*) \), then it is also achieved by the states that set

\[
(\mu_{0x}, \mu_{1x}) = \left( ((1 + a^* p^* - b^*(1 - p^*)) / 2, (1 - a^* p^* + b^*(1 - p^*)) / 2 \right)
\]

or

\[
(\mu_{0x}, \mu_{1x}) = \left( (1 - a^* p^* + b^*(1 - p^*)) / 2, (1 + a^* p^* - b^*(1 - p^*)) / 2 \right)
\]
for all \( x \). If Nature randomizes evenly over those two states, \( \delta^* \) is a best response, thus an equilibrium has been found. Intuitively, this equilibrium is characterized by both players ignoring the existence of \( X \). In particular, the value of this equilibrium (and hence of (5)) is \( R^B(N) \).

Consider now “misaligned” states \( s \) in which \( (\mu_{0x} - \mu_{1x})(\mu_{0x'} - \mu_{1x'}) < 0 \) for some \( x, x' \in X \). The constraint that \( |\mu_{1x} - \mu_{0x}| \leq \kappa \) for all \( x \) then implies that

\[
\mu_{1x} - \mu_{0x} + \mu_{0x'} - \mu_{1x'} = \mu_{1x} - \mu_{1x'} + \mu_{0x'} - \mu_{0x} \leq 2\kappa.
\]

Hence,

\[
2\kappa \geq \sup_{x \in X} \{\mu_{0x} - \mu_{1x}\} + \sup_{x \in X} \{\mu_{1x} - \mu_{0x}\} \geq a + b.
\]

All in all, the value of Nature’s best response problem can, therefore, be bounded above by the value of

\[
\max_{a, b, \pi \in [0, 1]^2: a + b \leq 2\kappa} \phi(a, b, p), \quad \phi(a, b, p) = pa \cdot f((1 - p)b - pa) + (1 - p)b \cdot [1 - f((1 - p)b - pa)]
\]

(6)

This is an upper bound because constraints on \( p \) are ignored – the optimal value of \( p \) may be inconsistent with \( \Sigma_X \)-measurability. The remainder of this proof will show that one of two cases must obtain: Either the value of (6) is attained by an aligned state, hence an aligned state best responds to \( \delta^* \), and \( (\delta^*, \pi^*) \) is an equilibrium; or the value of (6) can be bounded above by \( \kappa/2 \). If \( R^B(N) \geq \kappa/2 \), it again follows that an aligned state best responds to \( \delta^* \).

Suppose a solution \((a^*, b^*, p^*)\) to (6) is misaligned, then it immediately follows that \( p^* \in (0, 1) \).

Partial derivatives of \( \phi \) with respect to \( a \) and \( b \) are

\[
\phi_a = p \cdot f((1 - p)b - pa) - p(pa - (1 - p)b) \cdot f'((1 - p)b - pa)
\]

\[
= p [f((1 - p)b - pa) - (pa - (1 - p)b) \cdot f'((1 - p)b - pa)]
\]

\[
\phi_b = -(1 - p) \cdot f((1 - p)b - pa) + (1 - p)(pa - (1 - p)b) \cdot f'((1 - p)b - pa) + 1 - p
\]

\[
= (1 - p) [1 - f((1 - p)b - pa) + (pa - (1 - p)b) \cdot f'((1 - p)b - pa)]
\]

Recalling that \( f \) and \( f' \) are nonnegative and \( p^* \in (0, 1) \), it follows that \( p^*a - (1 - p^*)b \leq 0 \Rightarrow \phi_a > 0 \) and \( p^*a - (1 - p)b^* \geq 0 \Rightarrow \phi_b > 0 \). One of those always obtains, hence the constraint \( a + b \leq 2\kappa \) binds.

Substituting in for \( a = 2\kappa - b \) leads to the simplified problem

\[
\max_{b \in [0, 2\kappa], p \in [0, 1]} \tilde{\phi}(b, p), \quad \tilde{\phi}(b, p) = (2\kappa p - b) \cdot f(b - 2\kappa p) + (1 - p)b.
\]
Suppose \( b^* = 2\kappa \), thus \( a^* = 0 \). Then the value of the problem can be attained by setting \((a, b, p) = (0, 2\kappa p^*, 1)\), i.e. by an aligned state, so once again \((\delta^*, \pi^*)\) would be an equilibrium. A symmetric argument applies if \( b^* = 0 \). Suppose, therefore, that \( b^* \in (0, 2\kappa) \). This requires first-order conditions to hold. i.e.

\[
\begin{align*}
\tilde{\phi}_b &= -f(b - 2\kappa) + (2\kappa p - b) \cdot f'(b - 2\kappa) + 1 - p = 0, \\
\tilde{\phi}_p &= 2\kappa \cdot f(b - 2\kappa) - 2\kappa(2\kappa p - b) \cdot f'(b - 2\kappa) - b = 0.
\end{align*}
\]

Substituting into “\(-2\kappa \tilde{\phi}_b = \tilde{\phi}_p\)” and observing cancellations leads to \( b = 2\kappa(1 - p) \). Substituting this into the objective leads to a further concentrated objective function \( \bar{\phi}(p) = (4\kappa p - 2\kappa) \cdot f(2\kappa - 4\kappa p) + 2\kappa(1 - p)^2 \). Successively substituting the above simplifications into \( \Delta = b(1 - p) - a\pi \), one finds that \( \Delta = 2\kappa - 4\kappa p \), thus a change of variables toward \( \Delta \) as choice variable leads to the ultimately simplified problem

\[
\max_{\Delta \in [-2\kappa, 2\kappa]} \bar{\phi}(\Delta), \bar{\phi}(\Delta) = -\Delta f(\Delta) + \frac{(2\kappa + \Delta)^2}{8\kappa}. \tag{7}
\]

The objective function has derivatives

\[
\begin{align*}
\bar{\phi}_\Delta &= -\Delta f'(\Delta) - f(\Delta) + \frac{1}{2} + \frac{\Delta}{4\kappa}, \\
\bar{\phi}_{\Delta\Delta} &= -2f'(\Delta) - \Delta f''(\Delta) + \frac{1}{4\kappa}.
\end{align*}
\]

Recall that \( f(\Delta) = 1 - F((N - 1)/2; N, (1 + \Delta)/2) \), where \( F(n; N, \pi) \) denotes the binomial c.d.f. with indicated parameters. This firstly implies that \( f(0) = 1/2 \), thus inspection of \( \bar{\phi}_\Delta \) reveals a critical point at \( \Delta = 0 \) with value \( \bar{\phi}(0) = \kappa/2 \). Recalling furthermore that \( F(n; N, \pi) = (N - n)(n\pi) \int_0^{1 - \pi} t^{N - n - 1}(1 - t)^nt \) and hence

\[
\begin{align*}
f'(\Delta) &= (N + 1) \left( \frac{N}{(N - 1)/2} \right) 2^{-N-1}(1 - \Delta^2)^{N-1}/2, \\
f''(\Delta) &= -(N + 1) \left( \frac{N}{(N - 1)/2} \right) 2^{-N-1}(N - 1)\Delta(1 - \Delta^2)^{(N-3)/2},
\end{align*}
\]

one can write

\[
\bar{\phi}_{\Delta\Delta} = \frac{1}{4\kappa} - (N + 1) \left( \frac{N}{(N - 1)/2} \right) 2^{-N-1}(1 - \Delta^2)^{(N-3)/2} (2 - (N + 1) \Delta^2). \tag{8}
\]

This function is symmetric around \( \Delta = 0 \) and is increasing in \(|\Delta|\) whenever its value is negative. It follows that as \( \Delta \) ranges from \(-2\kappa\) to \(2\kappa\), \( \phi \) is either convex throughout, or concave throughout, or first convex, then concave, then convex. Thus if there is an interior optimum at all, then it coincides with the critical point identified at \( \Delta = 0 \), thus the value of an interior optimum would have to be \( \kappa/2 \).
Let \( \pi \) be the uniform distribution over states \( \{ s_i \}_{i=1}^{2^k} \), i.e. \( \pi_i \) assigns probability \( 2^{-i} \) to every \( s_i \). Notice the following features of \( \pi_i \): (i) The prior expectation of \( (Y_{0x}, Y_{1x}) \) equals \( (1/2, 1/2) \). (ii)

(ii) Consider the extended game in which the decision maker may also choose the sample design. If Nature plays \( \pi \), then it follows from the preceding paragraph’s observations that \( \sigma^B \) in conjunction with independent randomization is a best response. If DM chooses this strategy, then Nature’s best-response problem is just the one from part (i), hence \( \pi \) is a best response, establishing equilibrium. The same argument applies to the extended game in which the decision maker may also choose a sample design with respect to \( X \), and these arguments can also be combined.

(iii) Up to an obvious change in superscripts, the proof is the same up to expression (7). At this point, to verify crucial properties of \( \Theta(\Delta) \), evaluate its derivatives as before but recall that \( f(\Delta) = 1 - \Phi(0; \Delta, \sigma) = \Phi(0; -\Delta, \sigma) = \Phi(\Delta; 0, \sigma) \), where \( \Phi(\cdot; \mu, \sigma) \) stands for the normal c.d.f. with indicated parameters. It follows that

\[
\begin{align*}
f'(\Delta) &= (2\pi)^{-1/2} \sigma^{-1} \exp(-\Delta^2/(2\sigma^2)), \\
f''(\Delta) &= (2\pi)^{-1/2} \sigma^{-3} \Delta \exp(-\Delta^2/(2\sigma^2)),
\end{align*}
\]

thus one can write

\[
\Theta_{\Delta \Delta} = \frac{1}{4\kappa} - (2\pi)^{-1/2} \sigma^{-3} \exp(-\Delta^2/(2\sigma^2)) \cdot \left( \sigma^2 - 2\Delta^2 \right),
\]

which is amenable to the same analysis that was applied to (8).

Proposition 4 The core insight, shown in the next paragraph, is that if \( X \) is continuous, then the minimax regret value of the decision problem cannot be less than \( \kappa/2 \). Now assume the decision maker chooses \( \sigma^B \). Proposition 3 established that the value of Nature’s best response to \( \sigma^B \) is \( \max \{ R^B(N), \kappa/2 \} \). In those cases where the value equals \( R^B(N) \) (i.e. if \( \kappa \leq 2R^B(N) \)), proposition 1 established an equilibrium of the fictitious game and hence that \( \sigma^B \) achieves minimax regret. In those cases where the value is \( \kappa/2 \), \( \sigma^B \) achieves the lower bound on the decision problem’s minimax regret value and must, therefore, achieve minimax regret.

Using a quantile transformation, one can assume that \( X \) is distributed uniformly on \([0, 1] \). Mimicking Stoye (2009a, proposition 4), construct a sequence of priors \( \pi_i, i = 1, 2, \ldots \) as follows. Define the partition \( W_i \equiv \{ [0,1/i], (1/i, 2/i], \ldots, ((i-1)/i, 1] \} \) of the unit interval. Let \( (w_i^j)_{j=1}^{2^i} \) collect the subsets of \( W_i \) in arbitrary order. Define the collection of distributions \( \left( s_i^j \right)_{j=1}^{2^i} \) by identifying \( s_i^j \) with the degenerate distribution concentrated at

\[
(\mu_{0x}, \mu_{1x})_{x \in X} = \left( \begin{array}{c}
(1 + \kappa/2) 1 \{ x \in w_i^1 \} + (1 - \kappa/2) 1 \{ x \in w_i^2 \}, \\
(1 - \kappa/2) 1 \{ x \in w_i^1 \} + (1 + \kappa/2) 1 \{ x \in w_i^2 \}
\end{array} \right)_{x \in X}.
\]

Let \( \pi_i \) be the uniform distribution over states \( \left( s_i^j \right)_{j=1}^{2^i} \), i.e. \( \pi_i \) assigns probability \( 2^{-i} \) to every \( s_i^j \). Notice the following features of \( \pi_i \): (i) The prior expectation of \( (Y_{0x}, Y_{1x}) \) equals \( (1/2, 1/2) \). (ii)
With slight abuse of notation, let \( w_i(x) \) be the element of \( W_i \) that contains \( x \). Then \( s_x \) and \( s_{x'} \) are independent whenever \( w_i(x) \neq w_i(x') \). Now minimal adaptation of algebra from Stoye (2009a, proposition 4) shows that \( \lim_{i \to \infty} \min_{\delta \in D} \int R(\delta, s) \, d\pi_i = \kappa/2 \), thus \( \min_{\delta \in D} \sup_{S \subseteq S} R(\delta, s) \geq \kappa/2 \). The argument did not use the binomial sampling distribution, so it immediately extends to the Gaussian experiment.

**Proposition 5** It follows from proposition 3(i) that regret incurred by \( \delta' \) is bounded above by \( \max\{R^*(N), \kappa/2\} \). At the same time, Stoye (2009a) shows that the best no-data rule incurs maximal regret of \( 1/2 \). The claim follows immediately and again extends to the Gaussian experiment.

**Proposition 6** I show the claims for the binomial experiment, the argument for the Gaussian experiment is just the same.

(i) For any \( N_x \), let \( N'_x = \max\{M \leq N_x : M \text{ is odd}\} \). The least favorable prior that supports separated inference across covariates can be described as follows: For every \( x \in X \), \((\mu_{0x}, \mu_{1x})\) is equally likely to equal \(((1 + \Delta^*_x)/2, (1 - \Delta^*_x)/2)\) or \(((1 - \Delta^*_x)/2, (1 + \Delta^*_x)/2)\), where \( \Delta^*_x \) maximizes \( \Delta F \left( N'_x - \frac{1}{2}; N'_x, \frac{1+\Delta}{2} \right) \). For any \( x \neq x' \), \((\mu_{0x}, \mu_{1x})\) and \((\mu_{0x'}, \mu_{1x'})\) are realized independently. Then application of \( \delta^* \) separately across covariates is a best response for the decision maker. Yet if this is played, then additive separability of \( R(\cdot, s) \) in \( x \) ensures that the prior is indeed least favorable. (This is an abbreviated version of proposition 3 in Stoye (2009a).)

The claim is that for \( \min_{x \in X} N_x \) large enough, this prior is consistent with \( |\mu_{tx} - \mu_{tx'}| \leq \kappa \). To see it, observe that \( \max_{x,x' \in X, t \in \{0,1\}} |\mu_{tx} - \mu_{tx'}| \leq 2 \max_{x \in X} \Delta^*_{N_x} \). It therefore suffices to show that \( \Delta^*_{N_x} \to 0 \) as \( N_x \to \infty \). Now, substitute a normal approximation to the binomial c.d.f., it is easy to show that \( \sqrt{N_x \Delta^*_{N_x}} \to \arg \max \Delta \Phi(\Delta) \approx 0.75 \), where \( \Phi \) denotes the standard normal c.d.f.

(ii) The argument is similar, with some adjustment because \( N_x \) is now a random variable. The literature does not contain a formal statement for proposition 1 when \( N \) is random, but an easy extension of the proof shows that if \( N' = \max\{M \leq N : M \text{ is odd}\} \) is a random variable with distribution \( Q \), then the least favorable prior then randomizes equally over \( \{(1 + \Delta^*_N)/2, (1 - \Delta^*_N)/2\}, ((1 - \Delta^*_N)/2, (1 + \Delta^*_N)/2\} \), where \( \Delta^*_N = \arg \max \Delta \cdot \int F \left( N'_N - \frac{1}{2}; N'_N, \frac{1+\Delta}{2} \right) \, dQ \). Holding \( P(X) \) fixed and letting \( N \to \infty \), one finds that \( N_x/(N \cdot P(X = x)) \overset{P}{\to} 1 \). Substituting a normal approximation for the binomial c.d.f., one can then write \( \sqrt{N \Delta^*_{N_x}} \to \arg \max \Delta \Phi(\Delta \Pr(X = x)) \), thus \( \Delta^*_{N_x} \to 0 \).

**Proposition 7**
(i) With binary outcomes, $||P(Y_{tx}), P(Y_{tx'})||_{TV} = |\mu_{tx} - \mu_{tx'}|$, thus the claim restates proposition 3.

(ii) With binary outcomes, $||P(Y_{tx}), P(Y_{tx'})||_{LOR} \leq \gamma$ iff $\log \left( (\mu_{tx} - \mu_{tx'})/(\mu_{tx'} - \mu_{tx}) \right) \leq \gamma$. I will show that this, in turn, implies $|\mu_{tx} - \mu_{tx'}| \leq (\exp(\gamma/2) - 1) / (\exp(\gamma/2) + 1)$, which in turn implies the claim through proposition 2. While the implication $||P(Y_{tx}), P(Y_{tx'})||_{LOR} \leq \gamma \Rightarrow |\mu_{tx} - \mu_{tx'}| \leq (\exp(\gamma/2) - 1) / (\exp(\gamma/2) + 1)$ is one-sided, it will be seen that the result is tight.

Thus, consider the problem

$$\max_{(\mu, \rho) \in [0,1]^2} |\mu - \rho| \quad \text{s.t.} \quad \log \rho/(1-\mu) \leq \gamma.$$ 

Noting the problem’s symmetry, replace $|\mu - \rho|$ with $(\mu - \rho)$ in the objective for tractability. Letting $\lambda$ denote the Lagrange multiplier on the constraint, relevant partial derivatives are

$$\nabla_\mu (x) = 1 - \lambda \left( \frac{1}{1-x} + \frac{1}{1-x} \right)$$

$$\nabla_\rho (x) = -1 + \lambda \left( \frac{1}{1-x} + \frac{1}{1-x} \right).$$

Clearly $\nabla_\mu (x) = \nabla_\mu (1-x) = -\nabla_\rho (x) = -\nabla_\rho (1-x)$. As $\mu = \rho$ corresponds to minimizing the objective, a solution $(\mu^*, \rho^*)$ must have the feature that $\mu^* = 1 - \rho^*$. Using this constraint to eliminate $\rho$ and reparameterizing the problem by setting $\mu = (1+\Delta)/2 \Leftrightarrow \Delta = |\mu - \rho|$, one has the new problem

$$\max_{\Delta \in [0,1]} \Delta \quad \text{s.t.} \quad \gamma \geq \log \frac{(1+\Delta)^2}{(1-\Delta)^2} = 2 \log \frac{1+\Delta}{1-\Delta}.$$ 

The solution to this is characterized by the constraint binding, establishing the claim.

To see tightness of the bound for the case of two unknown treatments, note that a least favorable prior in proposition 3(i) has the feature that $\mu_{0x} = 1 - \mu_{1x}$, thus $(\mu_{0x}, \mu_{1x})$ from that prior solve the above problem for some $\gamma$. This last argument is not easily extended to the case of known $\mu_0$ however.

(iii) With binary outcomes, $D_{KL}(P(Y_{tx})||P(Y_{tx'})) \leq \gamma$ iff $\mu_{tx} \log(\mu_{tx}/\mu_{tx'}) + (1 - \mu_{tx}) \log((1 - \mu_{tx})/(1 - \mu_{tx'}))$. The argument is then similar to part (ii), except it is based on analyzing the maximization problem

$$\max_{(\mu, \rho) \in [0,1]^2} |\mu - \rho| \quad \text{s.t.} \quad \mu \log \frac{\mu}{\rho} + (1 - \mu) \log \frac{1 - \mu}{1 - \rho} \leq \gamma.$$ 

This problem has the same symmetry properties as the one in (ii), so a similar reparameterization leads to

$$\max_{\Delta \in [0,1]} \Delta \quad \text{s.t.} \quad \gamma \geq (1+\Delta/2) \log \frac{1+\Delta/2}{1-\Delta/2} + (1 - \Delta/2) \log \frac{1 - \Delta/2}{1 + \Delta/2} = \Delta \log \frac{1+\Delta/2}{1-\Delta/2}$$ 

and thus to the expression given in the proposition. Other remarks are as in (ii).
Example 1  Let $D \in \{0,1\}$ indicate assigned treatment, let $T \in \{0,1\}$ denote treatment that is in fact received, and let $Z \in \{00,01,11,10\}$ indicate a subject’s compliancy type (or principal stratum), where $Z = 00$ indicates never-takers, $Z = 01$ indicates compliers, and so on. Then one can write

$$
\begin{align*}
\mathbb{E}(Y|D = 1) &= \mathbb{E}(Y_1|D = 1, Z = 01) \Pr(Z = 01|D = 1) + \mathbb{E}(Y_0|D = 1, Z = 00) \Pr(Z = 00|D = 1) \\
&\quad + \mathbb{E}(Y_1|D = 1, Z = 11) \Pr(Z = 11|D = 1) + \mathbb{E}(Y_0|D = 1, Z = 10) \Pr(Z = 10|D = 1).
\end{align*}
$$

Treatment was assigned at random, thus $\mathbb{E}(Y_1|D = 1, Z = z) = \mathbb{E}(Y_1|Z = z)$ and $\Pr(Z = z|D = 1) = \Pr(Z = z)$, in particular $\Pr(Z = 01|D = 1) = 1 - \varepsilon$. Also using that $Y_0,Y_1 \in [0,1]$, one can then write

$$
\begin{align*}
\mathbb{E}(Y|D = 1) &\in [(1 - \varepsilon)\mathbb{E}(Y_1|Z = 01), (1 - \varepsilon)\mathbb{E}(Y_1|Z = 01) + \varepsilon] \\
\iff (1 - \varepsilon)\mathbb{E}(Y_0|Z = 01) &\in [\mathbb{E}(Y|D = 1) - \varepsilon, \mathbb{E}(Y|D = 1)].
\end{align*}
$$

Similar reasoning establishes that

$$(1 - \varepsilon)\mathbb{E}(Y_0|Z = 01) \in [\mathbb{E}(Y|D = 0) - \varepsilon, \mathbb{E}(Y|D = 0)].$$

Using that the ITE is $\Delta = \mathbb{E}(Y|D = 1) - \mathbb{E}(Y|D = 0)$, one can conclude

$$(1 - \varepsilon) (\mathbb{E}(Y_1|Z = 01) - \mathbb{E}(Y_0|Z = 01)) \in [\Delta - \varepsilon, \Delta + \varepsilon].$$

Now write

$$
\begin{align*}
\tilde{\Delta} &= \mathbb{E}Y_1 - \mathbb{E}Y_0 \\
&= (\mathbb{E}(Y_1|Z = 01) - \mathbb{E}(Y_0|Z = 01)) \Pr(Z = 01) + (\mathbb{E}(Y_1|Z \neq 01) - \mathbb{E}(Y_0|Z \neq 01)) \Pr(Z \neq 01) \\
&= (1 - \varepsilon) (\mathbb{E}(Y_1|Z = 01) - \mathbb{E}(Y_0|Z = 01)) + \varepsilon (\mathbb{E}(Y_1|Z \neq 01) - \mathbb{E}(Y_0|Z \neq 01)) \\
&\in [\Delta - 2\varepsilon, \Delta + 2\varepsilon].
\end{align*}
$$

Together with the trivial $\tilde{\Delta} \in [-1,1]$, this establishes validity of the bounds. They are achieved by the scenario indicated in the text, i.e. all noncompliers have $Z = 10$ and have individual treatment effects of either $+1$ or $-1$.

Example 2  The monotonicity assumption implies that $Z \in \{00,01,11\}$. With other variable labels as in example 1, elementary probability calculus yields

$$
\begin{align*}
\mathbb{E}(Y|D = 1) &= \mathbb{E}(Y_1|D = 1, Z \in \{01,11\}) \Pr(Z \in \{01,11\}|D = 1) + \mathbb{E}(Y_0|D = 1, Z = 00) \Pr(Z = 00|D = 1) \\
\mathbb{E}(Y|D = 0) &= \mathbb{E}(Y_0|D = 0, Z \in \{00,01\}) \Pr(Z \in \{00,01\}|D = 0) + \mathbb{E}(Y_1|D = 0, Z = 11) \Pr(Z = 11|D = 0).
\end{align*}
$$
Writing \( \varepsilon_0 = \Pr(Z = 0) \) and \( \varepsilon_1 = \Pr(Z = 1) \) and using independence of \( D \) and \( Z \), one can conclude

\[
\Delta = \mathbb{E}(Y|D = 1) - \mathbb{E}(Y|D = 0)
\]

\[
= (1 - \varepsilon_0) \mathbb{E}(Y_1|Z \in \{01, 11\}) + \varepsilon_0 \mathbb{E}(Y_0|Z = 00) - (1 - \varepsilon_1) \mathbb{E}(Y_0|Z \in \{00, 01\}) - \varepsilon_1 \mathbb{E}(Y_1|Z = 11),
\]

thus (using that \( Y_0, Y_1 \in [0, 1] \))

\[
(1 - \varepsilon_0) \mathbb{E}(Y_1|Z \in \{01, 11\}) - (1 - \varepsilon_1) \mathbb{E}(Y_0|Z \in \{00, 01\}) \in [\Delta - \varepsilon_0, \Delta + \varepsilon_1].
\]

Yet one can also write

\[
\hat{\Delta} = \mathbb{E}Y_1 - \mathbb{E}Y_0
\]

\[
= (1 - \varepsilon_0) \mathbb{E}(Y_1|Z \in \{01, 11\}) + \varepsilon_0 \mathbb{E}(Y_1|Z = 00) - (1 - \varepsilon_1) \mathbb{E}(Y_0|Z \in \{00, 01\}) - \varepsilon_1 \mathbb{E}(Y_0|Z = 11)
\]

\[
= \underbrace{(1 - \varepsilon_0) \mathbb{E}(Y_1|Z \in \{01, 11\}) - (1 - \varepsilon_1) \mathbb{E}(Y_0|Z \in \{00, 01\}) + \varepsilon_0 \mathbb{E}(Y_1|Z = 00) - \varepsilon_1 \mathbb{E}(Y_0|Z = 11)}_{\in [\Delta - \varepsilon_0, \Delta + \varepsilon_1] \text{ from above}}
\]

\[
\in \underbrace{[\Delta - \varepsilon_0 - \varepsilon_1, \Delta + \varepsilon_0 + \varepsilon_1]}_{\in [-\varepsilon_1, \varepsilon_0]} = [\Delta - \varepsilon_1, \Delta + \varepsilon_1].
\]

Together with \( \hat{\Delta} \in [-1, 1] \), this establishes validity of the bounds; that they are best possible is again easy to verify.

**Examples 3 and 4** Straightforward.

**Proposition 8**

(i) Let \( N \) be odd; the extension to \( N \) even is as before. The result does not extend to matched pairs or constrained randomization.

**Step 1: Describing the least favorable prior.** Every state of the world \( s \) implies expected values of latent outcomes in the sampling universe \((\mu_0, \mu_1)\) as well as true expected potential outcomes \((\bar{\mu}_0, \bar{\mu}_1)\), where \( \bar{\mu}_1 - \bar{\mu}_0 \in [a(\mu_1 - \mu_0) - b, a(\mu_1 - \mu_0) + b] \). It is easy to see that regret depends on the state of the world only through \((\mu_0, \mu_1, \bar{\mu}_0, \bar{\mu}_1)\), so states will be identified with these parameters. The least favorable prior \( \pi^* \) randomizes evenly over two states. The first of these is characterized by

\[
(\mu_0, \mu_1) = \left( \frac{1 - \Delta^*}{2}, \frac{1 + \Delta^*}{2} \right),
\]

implying that the ITE is \( \mu_1 - \mu_0 = \Delta^* \), where \( \Delta^* \geq 0 \) will be characterized in detail later. The ATE equals \( \Delta^* = \bar{\mu}_1 - \bar{\mu}_0 = a\Delta^* + b \), i.e. it achieves its upper bound. The second state in the support of \( \pi^* \) is generated by exchanging treatment labels; in particular, both \( \Delta^* \) and \( \bar{\Delta}^* \) just change signs (and the ATE achieves a lower bound).
Depending on parameter values, exactly one of two types of equilibrium obtains. Assume that the least favorable prior is as just described with $\Delta^* > 0$, then the planner’s unique best response is to play $\delta^B$, (the “equilibrium with learning”). Assume that $\Delta^* = 0$, then the sample data are noise, and any treatment rule is Bayes (the “noninformative equilibrium”).

Step 2: Equilibrium when $b \geq 1$. With $b > 1$, Nature can randomize evenly over two states s.t. $\Delta = 0$ in either state, thus the sample data are noise, but $\Delta = -1$ in one state and $\Delta = 1$ in the other. Clearly any decision rule incurs regret of $1/2$ against this prior. At the same time, the no-data rule $\delta^* = 1/2$ is easily seen to incur regret of at most $1/2$. For the remainder of the proof, assume $b < 1$.

Step 3: Simplifying Nature’s best-response problem. Nature’s best response must be supported on states $s$ that maximize

$$R(\delta^*, s) = \max \{ (\bar{\mu}_1 - \bar{\mu}_0)(1 - \mathbb{E}\delta^*(\omega)), (\bar{\mu}_0 - \bar{\mu}_1)\mathbb{E}\delta^*(\omega) \}.$$ 

Note that $\delta^*$ depends on $\omega$ only through $(I_{N-2}, I_N)$. Recalling that $(1 + I_N/N)/2$ is distributed as a binomial variable with parameters $(N, (1 + \mu_1 - \mu_0)/2)$, it follows that $\mathbb{E}\delta^*(\omega)$ depends on $(\mu_0, \mu_1)$ only through $\Delta \equiv \mu_1 - \mu_0$. Furthermore, a state $s$ solves Nature’s best-response problem if its implied values of $(\Delta, \tilde{\Delta})$ solve

$$\max_{\Delta \in [-1, 1]} \max_{\tilde{\Delta} \in [a\Delta - b, a\Delta + b] \cap [-1, 1]} \{ \tilde{\Delta}(1 - f(\Delta)), -\tilde{\Delta}f(\Delta) \},$$

where $f(\Delta) \equiv \mathbb{E}\delta^*(\omega)$. Noting that $f$ increases in $\Delta$ and $\tilde{\Delta}$ enters the objective linearly, this problem can be simplified to

$$\max_{\Delta \in [-1, 1]} \max \{ \min\{a\Delta + b, 1\}(1 - f(\Delta)), -\max\{a\Delta - b, -1\}f(\Delta) \}.$$ 

For a further simplification, note that $f(\Delta) = 1 - f(-\Delta)$ by the construction of $\delta$, thus $\min\{a(-\Delta) + b, 1\}(1 - f(-\Delta)) = \min\{-a\Delta + b, 1\}f(\Delta) = -\max\{a\Delta - b, -1\}f(\Delta)$. Hence, the problem is solved by some $\Delta^*$ iff it is solved by $-\Delta^*$ as well. With this in mind, there always exists a best response to $\delta^*$ that randomizes evenly over $(\Delta^*, \tilde{\Delta}^*)$ and $(-\Delta^*, -\tilde{\Delta}^*)$, where $\Delta^*$ solves

$$\max_{\Delta \in [-1, 1]} \{ \min\{a\Delta + b, 1\}(1 - f(\Delta)) \}$$

and $\tilde{\Delta}^* = \min\{a\Delta^* + b, 1\}$. Finally, this objective is negative whenever $\Delta < -b/a$ and decreases in $\Delta$ whenever $a\Delta + b > 1 \Leftrightarrow \Delta > (1 - b)/a$, thus the above expression can be further simplified to

$$\max_{\Delta \in [-b/a, (1-b)/a] \cap [-1, 1]} \phi(\Delta), \phi(\Delta) = (a\Delta + b)(1 - f(\Delta)),$$ 

the problem that will be considered henceforth. Note that as $a > 0$ and $b < 1$ are assumed, the feasible set for $\Delta$ (which will be suppressed below for brevity) always contains 0.
Step 4: Equilibrium when \( a \leq b \). In this case, \( \delta^*(\omega) = \frac{\omega}{b} \cdot \delta^B(\omega) + (1 - \frac{\omega}{b}) \cdot \frac{1}{2} \), thus \( f(\Delta) = \frac{1}{2} + \frac{\Delta}{2b} \), and (9) turns into

\[
\max_\Delta (a\Delta + b) \left( \frac{1}{2} - \frac{a}{2b} \Delta \right) = \max_\Delta \frac{1}{2} \left( b - \frac{a^2}{b} \Delta^2 \right),
\]

which is maximized by \( \Delta^* = 0 \), confirming the babbling equilibrium.

Step 5: Uniqueness of \( \Delta^* \). Let now \( a > b \). This step shows that (9) is solved by a unique \( \Delta^* \); furthermore, this \( \Delta^* \) has the same sign as \( \phi'(0) \) and in particular equals zero iff \( \phi'(0) = 0 \). For \( N = 1 \), direct evaluation of derivatives reveals that \( \phi \) is concave. For \( N \geq 3 \), write

\[
\phi''(\Delta) = -2af'(\Delta) - (a\Delta + b)f''(\Delta)
\]

and (recalling that the binomial c.d.f., \( F(n; N, p) = (N - n)C_n^{-N} \int_0^{1-p} t^{N-n-1}(1 - t)^n dt \))

\[
\begin{align*}
f(\Delta) &= 1 - \alpha \frac{N + 1}{2} \left( \frac{N}{2} \right) \sum_\varphi \int_0^{1-\Delta} t^{\frac{N-1}{2}} (1 - t)^{\frac{N-1}{2}} dt - (1 - \alpha) \frac{N - 1}{2} \left( \frac{N-2}{2} \right) \int_0^{1-\Delta} t^{\frac{N-3}{2}} (1 - t)^{\frac{N-3}{2}} dt, \\
f'(\Delta) &= -\alpha \varphi 2^{-N} (1 - \Delta^2)^{\frac{N-1}{2}} + (1 - \alpha) \psi 2^{-N} (1 - \Delta^2)^{\frac{N-3}{2}}, \\
f''(\Delta) &= -\alpha \varphi 2^{-N} (N - 1)\Delta (1 - \Delta^2)^{\frac{N-1}{2}} - (1 - \alpha) \psi 2^{-N} (N - 3)\Delta (1 - \Delta^2)^{\frac{N-3}{2}}.
\end{align*}
\]

Some algebra yields

\[
\phi''(\Delta) = 2^{-N} (1 - \Delta^2)^{\frac{N-5}{2}} \left[ \frac{-2a\alpha \varphi (1 - \Delta^2)^2 - 8a(1 - \alpha)\psi (1 - \Delta^2) + \alpha \varphi (N - 1)\Delta (1 - \Delta^2) (a\Delta + b) + 4(1 - \alpha)\psi (N - 3)\Delta (a\Delta + b)}{\gamma(\Delta)} \right].
\]

On \((-b/a, 1)\), the sign of this expression equals the sign of \( \gamma \). Using \( a\Delta + b \geq 0 \), \( \gamma \) is easily seen to be negative on \([\max\{-b/a, -1\}, 0]\). Furthermore, \( \gamma \) is a polynomial of fourth degree in \( \Delta \) with the following properties (using \( N \geq 3 \) and \( a\Delta + b \geq 0 \)): The coefficient on \( \Delta^4 \) is negative, thus \( \gamma \) is concave as \( \Delta \rightarrow \pm \infty \). The coefficient on \( \Delta^2 \) is positive, thus \( \gamma \) is convex at \( \Delta = 0 \). Thus, \( \gamma \) is first convex then concave on \([0, \infty)\). But is is also easily verified that \( \gamma(0) < 0 \) and \( \gamma(1) \geq 0 \), which is consistent with \( \gamma \) being first convex then concave on \([0, \infty)\) only if \( \gamma \) is first nonpositive then nonnegative on \([0, 1]\). It follows that \( \phi \) is first concave then convex on \([-b/a, 1]\). As also \( \phi(1) = 0 \), \( \phi \) is quasiconcave on the relevant range of \( \Delta \). As \( \phi \) is differentiable at 0, the claim follows.

Step 6: Developing the informative equilibrium. Evaluation of the combinatorial terms labeled \((\varphi, \psi)\) in step 4 reveals that

\[
\phi'(0) = \frac{a}{2} - bf'(0)
\]

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strictly decreases in $N^*$. Since $\alpha^*$ interpolates between neighboring $N^*$, $\phi'(0)$ can be thought of as a continuously decreasing function of $((N^* - 1)/2 + \alpha^*)$. Hence, if $\phi'(0) > 0$ for $(N^*, \alpha^*) = (1, 1)$, then $\phi'(0) > 0$ for $((N^* - 1)/2 + \alpha^*)$ below some threshold, $\phi'(0) = 0$ for $((N^* - 1)/2 + \alpha^*)$ on the threshold, and $\phi'(0) < 0$ for $((N^* - 1)/2 + \alpha^*)$ above the threshold. Let $N^*$ correspond to the threshold. If $N < N^*$, then existence of an informative equilibrium follows: The planner plays $\delta^*_N$, Nature’s best response is characterized by $\Delta^* \geq 0$, hence equilibrium obtains. If $N \geq N^*$, existence of a pooling equilibrium follows: The planner plays $\alpha^* \delta^*_{N^*} + (1 - \alpha^*) \delta^*_{N^* - 2}$, and Nature’s best response is characterized by $\Delta^* = 0$. It remains to verify whether $\phi'(0) > 0$ for $(N^*, \alpha^*) = (1, 1)$. Direct evaluation shows that this is the case iff $a \geq b$.

(ii) Follows as in proposition 1.

(iii) The equilibrium supporting this minimax regret treatment rule is very similar to the one from part (i). (Note in particular that the probabilistic transformation is equivalent to adding a mean-preserving spread to the signal and then applying $\delta^*$, thus it can be thought of as smooth adjustment of sample size.) Steps 1 through 4 of the proof go through unchanged. Step 5 is unchanged up to (10). To continue from there, write

$$f(\Delta) = \Phi(\Delta; 0, 2\sigma),$$
$$f'(\Delta) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp(-\Delta^2/2\sigma^2),$$
$$f''(\Delta) = -\frac{1}{\sqrt{2\pi\sigma^2}} \frac{\Delta}{\sigma^2} \exp(-\Delta^2/2\sigma^2)$$

and therefore (after algebra)

$$\phi''(\Delta) = \frac{\exp(-\Delta^2/2\sigma^2)}{\sqrt{2\pi\sigma^2}} \left( a \sigma^2 \Delta^2 + \frac{b}{\sigma^2} \Delta - 2a \right).$$

Using standard rules for analyzing quadratic equations, $\phi''(\Delta)$ is nonpositive on

$$\left[ -\frac{b}{2a} - \sqrt{\frac{b^2}{4a^2} + 2\sigma^2}, -\frac{b}{2a} + \sqrt{\frac{b^2}{4a^2} + 2\sigma^2} \right],$$

an interval that in particular contains $[-b/a, 0]$, and nonnegative otherwise. Step 5 can then be completed as before.

Regarding step 6, $f'(0)$ continuously decreases in $\sigma$ and approaches 0 as $\sigma \to \infty$. Thus, if $\phi'(0) = \frac{a}{\sigma} - b \phi'(0) < 0$, then the first-order condition $\phi'(0) = 0$ can be achieved by increasing $\sigma$. Specifically, simple algebra shows that

$$\phi'(0) = 0 \iff \sigma = \frac{b}{a} \sqrt{\frac{2}{\pi}}.$$
If \( \sigma \) is smaller than this threshold, the decision maker can increase it artificially by replacing \( \omega \) with \( \omega + \xi \), where \( \xi \sim N \left( 0, \frac{2}{\pi} \right) \) and then applying \( \delta^G \). This is equivalent to the rule given in the proposition.

**Corollary 6** Follows by evaluating (9).

**Corollary 7** Equilibrium is informative, and the decision rule therefore \( \delta^* \), if \( \phi'(0) \geq 0 \) for \( (N^*, \alpha^*) = (N, 1) \). Algebraic evaluation of this condition leads to the corollary.

**Proposition 8** The proof strategy is similar as before, I only point out necessary adaptations.

**Step 1: Describing the least favorable prior.** As before, except that the algebraic expression for the bounds on \( \Delta \) in terms of \( \Delta \) changed.

**Step 2: Simplifying Nature’s best-response problem.** Let \((\mu_0, \mu_1, \tilde{\mu}_0, \tilde{\mu}_1) = (\mathbb{E}(Y_0|Z=1), \mathbb{E}(Y_1|Z=1), \mathbb{E}(Y_0), \mathbb{E}(Y_1))\), then it is easy to see that \( R(\delta^*, s) \) depends on \( s \) only through these quantities, and Nature’s best-response problem boils down to

\[
\max_{(\mu_0, \mu_1) \in [0,1]^2} \max \{ (\tilde{\mu}_1 - \tilde{\mu}_0) (1 - f(\Delta)), (\tilde{\mu}_0 - \tilde{\mu}_1) f(\Delta) \}
\]

s.t.
\[
\frac{\mu_0}{\exp(c) + (1 - \exp(c))\mu_0} \leq \tilde{\mu}_0 \leq \frac{\exp(c)\mu_0}{1 + (\exp(c) - 1)\mu_0}, \\
\frac{\mu_1}{\exp(c) + (1 - \exp(c))\mu_1} \leq \tilde{\mu}_1 \leq \frac{\exp(c)\mu_1}{1 + (\exp(c) - 1)\mu_1},
\]

where \( \Delta = \mu_1 - \mu_0 \) as before. Inspection of the way in which \((\tilde{\mu}_0, \tilde{\mu}_1)\) enter the objective reveals immediate simplification to

\[
\max_{(\mu_0, \mu_1) \in [0,1]^2} \max \left\{ \left( \frac{\exp(c)\mu_0}{1 + (\exp(c) - 1)\mu_1} - \frac{\mu_0}{\exp(c) + (1 - \exp(c))\mu_0} \right) (1 - f(\Delta)), \right\}.
\]

This objective does \textit{not} depend on \((\mu_0, \mu_1)\) only through \( \Delta \). However, evaluation of first and second derivatives reveals that the problem

\[
\max_{(\mu_0, \mu_1) \in [0,1]^2: \mu_1 = \mu_0 + \Delta} \left\{ \frac{\exp(c)\mu_1}{1 + (\exp(c) - 1)\mu_1} - \frac{\mu_0}{\exp(c) + (1 - \exp(c))\mu_0} \right\}
\]

is solved by \( \mu_0 = (1 - \Delta)/2 = 1 - \mu_1 \), and similarly for \( \left( \frac{\exp(c)\mu_0}{1 + (\exp(c) - 1)\mu_1} - \frac{\mu_1}{\exp(c) + (1 - \exp(c))\mu_0} \right) \). It follows that a best response for Nature must have \( \mu_0 = 1 - \mu_1 \), in which case the maximization problem can
be further simplified to
\[
\max_{\Delta \in [-1, 1]} \max_{\rho(\Delta)} \left\{ \frac{(1 + \Delta) \exp(c) - (1 - \Delta)}{(1 + \Delta) \exp(c) + (1 - \Delta)} (1 - f(\Delta)), \frac{1 + \Delta - (1 - \Delta) \exp(c)}{1 + \Delta + (1 - \Delta) \exp(c)} f(\Delta) \right\}.
\]

This objective exhibits the same symmetry as before, thus a least favorable prior can be constructed as before. Also taking into account that \(\rho(\Delta) < 0\) if \(\Delta < (1 - \exp c)/(1 + \exp c)\), whereas \(\rho(\Delta) < 1\) for any \(\Delta < 1\), \(\Delta^*\) must solve
\[
\max_{\Delta \in [(1 - \exp c)/(1 + \exp c), 1]} \phi(\Delta) = \rho(\Delta) (1 - f(\Delta)), \tag{11}
\]
the problem that will be analyzed henceforth.

**Step 3: Equilibrium when** \(c \geq \ln(2 + \sqrt{3})\). In this case, \(f(\Delta) = \frac{1}{2} - \frac{2 \exp c}{\exp(2c) - 1} \Delta\), thus (11) simplifies to
\[
\max_{\Delta} \rho(\Delta) \left( \frac{1}{2} - \frac{2 \exp c}{\exp(2c) - 1} \Delta \right).
\]
By evaluating derivatives, this can be verified to be solved by \(\Delta^* = 0\) as required. (Readers who wish to verify algebra should take note that \(\rho'(\Delta) = 4 \exp c/((1 + \Delta) \exp c + (1 - \Delta))^2\).

**Step 4: Uniqueness of \(\Delta^*\).** The argument for quasiconcavity of \(\phi\) becomes more complex, the remainder is as before. For \(N = 1\), one can verify concavity of \(\phi\) by evaluating derivatives. For \(N \geq 3\), observe first that \(\rho\) is increasing and concave, whereas algebra from the preceding proof revealed that \(1 - f(\Delta)\) is decreasing and concave, over \(\Delta \in [-1, 0]\). These facts jointly imply that \(\phi\) is concave on \([(1 - \exp c)/(1 + \exp c), 0]\). It is also easily verified that \(\phi(0) = \frac{\exp c - 1}{2(\exp c + 1)} > 0\) and that \(\phi(1) = 0\). It follows that \(\phi(\Delta)\) has a critical point \(\Delta_0\) on \([0, 1]\). Let \(k = \rho(\Delta_0)\) and \(l = \rho'(\Delta_0)\), then \(\rho_0(\Delta) = \rho(\Delta_0) + \rho'(\Delta_0)(\Delta - \Delta_0)\) is the tangent to \(\rho\) at \(\Delta_0\). As \(\rho\) is concave with positive intercept, \(\rho_0\) is positive affine, thus \(\phi(\Delta) \leq \rho_0(\Delta)(1 - f(\Delta))\). But with \(\rho_0\) being positive affine, algebra from step 4 of the preceding proof can be entirely mimicked to show that \(\rho_0(\Delta)(1 - f(\Delta))\) is quasiconcave on \([0, 1]\), is positive and increasing at 0, and equals 0 at 1. It follows that \(\rho_0(1 - f)\) has at most one maximum on \([0, 1]\), and any such maximum obtains iff a first-order condition holds. But \(\frac{d}{d\Delta} \rho_0(\Delta_0)(1 - f(\Delta_0)) = \rho_0(\Delta_0)(-f'(\Delta_0)) + \rho_0'(\Delta_0)(1 - f(\Delta_0)) = \rho_0(\Delta_0)(-f'(\Delta_0)) + \rho'(\Delta_0)(1 - f(\Delta_0)) = 0\), so the maximum occurs at \(\Delta_0\). Since also \(\rho_0(\Delta_0)(1 - f(\Delta_0)) = \rho(\Delta_0)(1 - f(\Delta_0))\), it follows that \(\phi(\Delta)\) is uniquely maximized at \(\Delta_0\).

**Step 5: Developing the informative equilibrium.** This is essentially as before. The crucial condition for \(\phi'(0) \geq 0\) at \((N^*, \alpha^*) = (1, 1)\) now is
\[
\rho'(0) - \rho(0) \geq 0,
\]
which can be verified to obtain \( c \leq \ln (2 + \sqrt{3}) \).

**Corollary 9** Follows by evaluating (11).

**Corollary 10** The condition for equilibrium to be informative now is

\[
\phi'(0) = \frac{\phi''(0)}{2} - \rho(0)f'(0) \geq 0 \iff f''(0) \leq 2 \exp c/(\exp 2c - 1).
\]

**References**


