# Microeconomics Comprehensive Exam

August 2010

Instructions:
(1) Please answer each of the four questions on separate pieces of paper.
(2) Please write only on one side of a sheet of paper
(3) When finished, please arrange your answers alphabetically
(in the order in which they appeared in the questions, i.e. 1 (a), 1 (b), etc.).

**1.** An individual has a continuous increasing Bernoulli utility function $u(\cdot)$ and initial wealth $w$. Let lottery $L$ offer a payoff of $G$ with probability $\pi$, and payoff of $B$ with probability $1 - \pi$.

(a) If the individual owns the lottery, what is the minimum price she would sell it for?

(b) If the individual does not own the lottery, what is the maximum price she would buy it for?

(c) Are buying and selling prices equal? Give an economic interpretation for your answer. Find an example of preferences for which buying and selling prices are equal.

(d) Let $G = 10$, $B = 5$, $w = 10$, $\pi = 1/2$, and $u(x) = \sqrt{x}$. Compute the buying and selling prices for this lottery and this utility function.

**2.** Suppose that the two consumers have utility functions

$$u_1(x) = \frac{1}{2}\log(x_1^1 + x_2^1) + \frac{1}{2}\log(x_1^2)$$

and

$$u_2(x) = \frac{1}{2}\log(x_2^1) + \frac{1}{2}\log(x_2^2)$$

(where the superscript denotes the good and the subscript denotes the consumer), and endowments $e_1 = (1, 2)$ and $e_2 = (2, 1)$, respectively.

(a) Give programming characterization of Pareto optimal allocations for this economy (ignoring initial endowments).

(b) Carefully define a competitive equilibrium for this economy.

(c) Characterize and compute competitive equilibria (accounting for initial endowments).

(d) From these characterizations what can you deduce about the optimality of competitive allocations? Use a diagram to justify your claim.

**3.** Two generals approach an enemy city from two different directions. The city fortifications are strong with probability $\varepsilon \in (0,1)$ and weak with probability $1-\varepsilon$. Along his approach route, General Primus has a direct view into the city. He observes whether the city fortifications are weak or strong. General Secundus cannot see the quality of the fortifications.

During the night before attack, the two generals attempt to communicate with each other. The communication develops in the following way. If the fortifications are strong, nothing happens. If Primus observes that the fortifications are weak, he sends a messenger to general Secundus. The messenger's route is dangerously close to the city. With probability $\alpha \in (0,1)$, the messenger is caught and killed, and with probability $1-\alpha$, the messenger reaches general Secundus and informs the general that the city fortifications are weak. In order to make sure that Primus knows that the messenger passed safely, general Secundus sends the messenger back. On the way back, the messenger gets caught with probability $\alpha$ and with the remaining probability $1-\alpha$, the messenger reaches general Primus safely. In order to let Secundus know that the messenger reached safely, Primus sends the messenger back again. When the messenger reaches Secundus (with probability $1-\alpha$), he is sent back again. Back and forth, as long as the messenger reaches one of the generals alive, he is immediately sent back. Eventually, the messenger gets caught with probability 1.

We represent the information structure as a type space. Let $t_i$, where $i = 1, 2$, be the number of times that general $i$ sends the messenger away. We refer to $t_i$ as the type of general $i$.Then, $t_i$ is finite with probability 1 and the joint probability of a pair of types $(t_1, t_2)$ is equal to

$$\begin{aligned} \text{Prob}\,(t_1 = 0, t_2 = 0) &= \varepsilon, \text{ and for each } t \geq 1 \\ \text{Prob}\,(t_1 = t, t_2 = t-1) &= (1-\varepsilon)\,\alpha\,(1-\alpha)^{2t-1}\,, \\ \text{Prob}\,(t_1 = t, t_2 = t) &= (1-\varepsilon)\,\alpha\,(1-\alpha)^{2t}\,, \end{aligned}$$

and $\text{Prob}(t_1 = m_1, t_2 = m_2) = 0$ for any other pair $m_1$ and $m_2$. The fortifications are strong if and only if $t_1 = 0$. For example, type $t_1 = 0$ is the type of general Primus who observes that the fortifications are strong; type $t_2 = 0$ is a type of Secundus who observes no messenger (and as a result assigns positive probability to each of two events: either (a) that fortifications are strong and no messenger was sent or (b) that the fortifications are weak, the messenger was sent once, and he get caught along the way).

In the morning, each general separately chooses whether to attack the city. The payoffs depend on whether the city fortifications are weak or strong and they are represented in the following tables:

Weak

| G1/G2 | A | D |
|---|---|---|
| A | 1,1 | -2,0 |
| D | 0,-2 | 0,0 |

and

Strong

| G1/G2 | A | D |
|---|---|---|
| A | -2,-2 | -2,0 |
| D | 0,-2 | 0,0 |

In other words, the attack is successful if and only if the two generals attack simultaneously and the fortifications are weak. In the case of successful attack, both generals receive payoff 1; if the attack fails, the general who attacks receives payoff $-2$ and the general who does not attack receives payoff 0.

(a) State the definition of Bayesian Nash equilibrium in games with incomplete information.

(b) Show that there always exists an equilibrium in which none of the generals attacks, no matter what is their type.

(c) Suppose that $\alpha \geq \frac{2\varepsilon}{1-\varepsilon}$. Show that there exists an equilibrium, in which general Secundus (but not Primus) always attacks.

(d) Suppose that $\alpha < \frac{2\varepsilon}{1-\varepsilon}$. Show that the equilibrium in which no general attacks is unique (i.e., there is no other equilibrium).

**4.** (a) There are $n$ identical objects and $m$ individuals, where $m > n$. Any individual can get at most one unit of the object. Let $\Theta_i = \mathbb{R}_{++}$ be the set of $i$'s possible valuations.
Preferences are assumed to be quasi-linear in the following form,

$$u_i(g_i, t_i; \theta_i) = \theta_i g_i + t_i,$$

where $g_i \in \{0, 1\}$ denotes the allocation of the objects ($g_i = 1$ if $i$ gets one, $g_i = 0$ otherwise) and $t_i$ denotes income transfers to $i$. Let $f = (g, t)$ be a social choice function, where $g : \prod_{i=1}^{m} \Theta_i \to \{0, 1\}^m$ and $t : \prod_{i=1}^{m} \Theta_i \to \mathbb{R}^m$ satisfy $\sum_{i=1}^{m} t_i(\theta) \leq 0$.

(a)-(i) Calculate the social choice function which is prescribed by the pivotal mechanism.
(a)-(ii) Verify that it is always a (weakly) dominant strategy for each individual to submit his true valuation.

— You may ignore ties in the argument.

(b) Consider that a single object is gong to be sold via the first-price auction, in an environment with symmetrically and independently distributed private values. There are $n$ risk-averse bidders with identical von-Neumann-Morgenstern index $v : \mathbb{R} \to \mathbb{R}$ with $v(0) = 0$, whose valuations are drawn from $[0, \overline{\theta}]$ according to an i.i.d. differentiable distribution $F$ with density $f$.
Derive the differential equation which the bidding function in symmetric Bayesian Nash equilibrium must satisfy.

(c) State and prove the revelation principle for implementation in Bayesian Nash equilibrium.