STOCHASTIC POLICY DESIGN IN A LEARNING ENVIRONMENT WITH RATIONAL EXPECTATIONS

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Abstract. In this paper we present a method for using rational expectations in a stochastic linear-quadratic optimization framework in which the unknown parameters are updated through a learning scheme. We use the QZ decomposition as suggested by Sims (1996) to solve the rational expectations part of the model. The parameter updating is done with the Kalman filter and the optimal control is calculated using the covariance matrix of the uncertain parameter.

1. Introduction

There has been a recent revival of interest in learning under the title of bounded rationality, Marcet and Sargent (1989) and Sargent (1993). Earlier works on learning in macroeconomics includes studies by Prescott (1972), MacRea (1972), Chow (1975) and Kendrick (1981). In two recent papers, Amman and Kendrick (1998a, 1998c), we employed the Sims QZ decomposition approach to solve a rational expectations model in a deterministic optimal control context with parameter updating using the Ljung and Söderström (1983) self tuning regulator. Here we extend that work to the case where the parameters are unknown and treated as stochastic and the optimal instruments are computed while taking into account the variance and covariances of the parameter estimates. The learning of the parameters is done through the use of a Kalman filter.

2. Problem statement

Following Kendrick (1981), the standard single-agent stochastic linear-quadratic (LQ) optimization problem is written as:

Find the set of admissible instruments $U = \{u_0, u_1, ..., u_{T-1}\}$ that minimizes the welfare loss function

$$J_T = E\left\{\beta^T L_T(x_T) + \sum_{t=0}^{T-1} \beta^t L_t(x_t, u_t)\right\} \tag{1}$$

with

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$$L_T = \frac{1}{2}(x_T - \bar{x}_T)'W(x_T - \bar{x}_T)$$
$$L_t = \frac{1}{2}(x_t - \bar{x}_t)'W(x_t - \bar{x}_t) + \frac{1}{2}(u_t - \bar{u}_t)'R(u_t - \bar{u}_t) + (x_t - \bar{x}_t)'F(u_t - \bar{u}_t)$$

subject to the model

$$x_{t+1} = A(\theta)x_t + B(\theta)u_t + c(\theta)_t + \epsilon_t \tag{2}$$

The vector $x_t \in \Re^n$ is the state of the economy at time t and the vector $u_t \in \Re^m$ contains the policy instruments. The initial state of the economy $x_0 \in \Re^n$ is known, $\bar{x}_t \in \Re^n$ and $\bar{u}_t \in \Re^m$ are target values. $W \in \Re^{(n\times n)}$, $R \in \Re^{(m\times m)}$ and $F \in \Re^{(n\times m)}$ are penalty matrices; $\epsilon_t \in \Re^n$ is a white noise vector with $\epsilon_t \sim N(0, \Sigma^{\epsilon\epsilon})$. We assume that $\Sigma^{\epsilon\epsilon} \in \Re^{(n\times n)}$ is known to the policy maker. Learning is introduced into the LQ framework by the unknown parameter vector $\theta \in \Re^p$ which is determined through a learning strategy.

The above model is straightforward to solve, see Kendrick (1981). However, a serious drawback for economics is that equation (2) does not allow for RE. One way of allowing RE to enter the model is to augment equation (2) in the following fashion

$$x_{t+1} = A(\theta)x_t + B(\theta)u_t + c(\theta)_t + \sum_{j=1}^{k} D_j(\theta)E_t x_{t+j} + \epsilon_t \tag{2a}$$

where the matrix $D_j(\theta)$ is a parameter matrix, $E_t x_{t+j}$ is the expected state for time $t+j$ as seen from time t, and k is the maximum lead in the expectations formation[1].

In order to compute the admissible set of instruments we have to eliminate the rational expectations from the model. In an earlier paper, Amman and Kendrick (1998a), we described how the control model with RE can be solved using Sims (1996) approach. In this case we employ this method for the situations in which there is learning.

3. Solving Rational Expectations

For the sake of simplicity let us assume that we have an estimate of the parameter vector $\hat{\theta}_{t|t}$. In our notations this is the estimated value of θ at time t using the observations through time t. The covariance matrix of this parameter vector is defined as $\hat{\Sigma}^{\theta\theta}_{t|t}$. For the time being we will treat $\hat{\theta}_{t|t}$ as being constant. In a later phase we return to the issue of reestimating $\hat{\theta}_{t|t}$.

In the last decade, a number of generic methods to solve models with rational expectations were developed. For instance, Fair and Taylor (1983) use an iterative method for solving RE models and, in the tradition of Theil (1968), Fisher, Holly and Hughes Hallett (1986) use a method based on stacking the model variables. McCallum (1983) and Uhlig (1997) use the method of undetermined coefficients. A hybrid method based

[1] See also Amman (1996).

on the saddle point property is presented in Anderson and Moore (1985).

Recently, Sims (1996) proposed a method based on the QZ decomposition. Following Sims (1996), which is an extension the work of Blanchard and Kahn (1980), we can rewrite the system equation (2a) in the following augmented form

$$\Gamma_0 \tilde{x}_{t+1} = \Gamma_1 \tilde{x}_t + \Gamma_2 u_t + \Gamma_{3,t} + \Gamma_4 \epsilon_t \tag{3}$$

where

$$\Gamma_0 = \begin{bmatrix} I - D_1(\hat{\theta}_{t|t}) & -D_2(\hat{\theta}_{t|t}) & \dots & -D_{k-1}(\hat{\theta}_{t|t}) & -D_k(\hat{\theta}_{t|t}) \\ I & 0 & \dots & 0 & 0 \\ 0 & I & \dots & 0 & 0 \\ \vdots & & \ddots & 0 & 0 \\ 0 & \dots & & I & 0 \end{bmatrix} \tag{4}$$

$$\Gamma_1 = \begin{bmatrix} A(\hat{\theta}_{t|t}) & 0 & 0 & \dots & 0 \\ 0 & I & 0 & \dots & 0 \\ 0 & 0 & I & & 0 \\ \vdots & \vdots & & \ddots & \\ 0 & 0 & 0 & & I \end{bmatrix} \quad \Gamma_2 = \begin{bmatrix} B(\hat{\theta}_{t|t}) \\ 0 \\ \vdots \\ 0 \end{bmatrix} \quad \Gamma_{3,t} = \begin{bmatrix} c(\hat{\theta}_{t|t})_t \\ 0 \\ \vdots \\ 0 \end{bmatrix} \quad \Gamma_4 = \begin{bmatrix} I \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

and the augmented state vector

$$\tilde{x}_t = \begin{bmatrix} x_t \\ Ex_{t+1} \\ Ex_{t+2} \\ \vdots \\ Ex_{t+k-1} \end{bmatrix} \tag{5}$$

Taking the generalized eigenvalues of equation (3) allows us to decompose the system matrices Γ_0 and Γ_1 in the following manner, see Moler and Stewart (1973) or Coleman and Van Loan (1988),

$$\Lambda = Q\Gamma_0 Z$$
$$\Omega = Q\Gamma_1 Z$$

with $Z'Z = I$ and $Q'Q = I$. The matrices Λ and Ω are upper triangular matrices and the generalized eigen values are $\forall i \ \omega_{i,i}/\lambda_{i,i}$. If we use the transformation $w_t = Z'\tilde{x}_t$ we can write equation (3) as

en t-5(h)10(e)-339(t-5ro)12im

$$\Lambda w_{t+1} = \Omega_1 w_t + Q\Gamma_2 u_t + Q\Gamma_{3,t} + Q\Gamma_4 \epsilon_t \tag{6}$$

$$\begin{bmatrix}\Lambda_{11} & \Lambda_{12}\\ 0 & \Lambda_{22}\end{bmatrix}\begin{bmatrix}w_{1,t+1}\\ w_{2,t+1}\end{bmatrix} = \begin{bmatrix}\Omega_{11} & \Omega_{12}\\ 0 & \Omega_{22}\end{bmatrix}\begin{bmatrix}w_{1,t}\\ w_{2,t}\end{bmatrix} + \begin{bmatrix}Q_1\\ Q_2\end{bmatrix}\Gamma_2 u_t + \begin{bmatrix}Q_1\\ Q_2\end{bmatrix}\Gamma_{3,t} + \begin{bmatrix}Q_1\\ Q_2\end{bmatrix}\Gamma_4\epsilon_t \tag{7}$$

where the unstable eigenvalues are the in lower right corner, i.e in the matrices Λ_{22} and Ω_{22}. By forward propagation and taking expectations, it is possible to derive $w_{2,t}$ as a function of future instruments and exogenous variables

$$\gamma_t = w_{2,t} = -\sum_{j=0}^{\infty} M^{j-1}\Omega_{22}^{-1}Q_2(\Gamma_2 u_{t+j} + \Gamma_{3,t}) \tag{8}$$

with

$$M = \Omega_{22}^{-1}\Lambda_{22}$$

Reinserting equation (8) into equation (6) gives us

$$\tilde{\Lambda} w_{t+1} = \tilde{\Omega} w_t + \tilde{\Gamma}_2 u_t + \tilde{\Gamma}_{3,t} + \tilde{\Gamma}_4 \epsilon_t + \tilde{\gamma}_t \tag{9}$$

with

$$\tilde{\Lambda} = \begin{bmatrix}\Lambda_{11} & \Lambda_{12}\\ 0 & I\end{bmatrix} \quad \tilde{\Omega} = \begin{bmatrix}\Omega_{11} & \Omega_{12}\\ 0 & 0\end{bmatrix} \quad \tilde{\Gamma}_2 = \begin{bmatrix}Q_1\\ 0\end{bmatrix}\Gamma_2$$

$$\tilde{\Gamma}_{3,t} = \begin{bmatrix}Q_1\\ 0\end{bmatrix}\Gamma_{3,t} \qquad \tilde{\Gamma}_4 = \begin{bmatrix}Q_1\\ 0\end{bmatrix}\Gamma_4 \qquad \tilde{\gamma}_t = \begin{bmatrix}0\\ \gamma_t\end{bmatrix}$$

Knowing that $\tilde{x}_t = Z' w_t$ we can write equation (9) as

$$\tilde{x}_{t+1} = \tilde{A}\tilde{x}_t + \tilde{B} u_t + \tilde{c}_t$$

4. Computing the stochastic optimal solution

Now we have the model in the form of equation (10) we can derive the optimal solution of the model in equations (1)-(2a). The optimal solution can be obtained through solving the so called Riccati equation and tracking equation backward in time[2]

$$\begin{aligned} K_t &= \tilde{W} + \beta E\{\tilde{A}' K_{t+1} \tilde{A}\} - [\tilde{F} + \beta E\{\tilde{A}' K_{t+1} \tilde{B}\}] \times \\ & [R + \beta E\{\tilde{B}' K_{t+1} \tilde{B}\}]^{-1} [\beta E\{\tilde{B}' K_{t+1} \tilde{A}\} + \tilde{F}] \end{aligned} \quad (11)$$

$$\begin{aligned} p_t &= \beta E\{\tilde{A}' K_{t+1} \tilde{c}_t\} + \beta E\{\tilde{A}'\} p_{t+1} - [\tilde{F} + \beta E\{\tilde{A}' K_{t+1} \tilde{B}\}] \times \\ & [R + \beta E\{\tilde{B}' K_{t+1} \tilde{B}\}]^{-1} [\beta E\{\tilde{B}' K_{t+1} \tilde{c}_t + \beta E\{\tilde{B}'\} p_{t+1}] \end{aligned} \quad (12)$$

with the boundary conditions

$$\begin{aligned} K_T &= \tilde{W} \\ p_T &= -\tilde{W} \bar{x}_T \end{aligned}$$

The penalty matrices $\tilde{W}$ and $\tilde{F}$ are the penalty matrices from the objective function adjusted to conformable size. Once we have backward integrated these equations and the fact that x_0 is known, we can compute the set optimal instruments by forward integrating

$$u_t = G_t \tilde{x}_t + g_t \quad (13)$$

with systems equation (10) where

$$\begin{aligned} G_t &= -[R' + \beta E\{\tilde{B}' K_{t+1} \tilde{B}\}]^{-1} [\tilde{F}' + \beta E\{\tilde{B}' K_{t+1} \tilde{A}\}] \quad (14) \\ g_t &= -[R' + \beta E\{\tilde{B}' K_{t+1} \tilde{B}\}]^{-1} [\beta E\{\tilde{B}' K_{t+1} \tilde{c}_t + \beta E\{\tilde{B}'\} p_{t+1}] \quad (15) \end{aligned}$$

The above equations allows us to solve the set of admissible instruments. The components like $E\{\tilde{A}' K_{t+1} \tilde{B}\}$ capture the effect of the parameter uncertainty on the value instruments. The elements of these components, Magnus and Neudecker (1994), are

$$d_{i,j} = \hat{\tilde{a}}_i' K_{t+1} \hat{\tilde{b}}_j + tr(K_{t+1} \Sigma^{\tilde{b}_j \tilde{a}_i}) \quad (16)$$

where $\hat{\tilde{a}}_i$ is the expected i column of the matrix $\tilde{A}$ and expected $\hat{\tilde{b}}_j$ is the expected j column of the matrix $\tilde{B}$; $tr()$ is the trace operator. Hence, through the covariance matrix $\Sigma^{\tilde{b}_j \tilde{a}_i}$ the uncertainty of the parameters on the instruments is captured.

5. The learning algorithm

As mentioned earlier, the components $\tilde{A}$, $\tilde{B}$, $\tilde{c}_t$ may depend on the unknown parameter vector θ and we have inserted an estimate $\hat{\theta}_{t|t}$ of this parameter vector in order to be able to solve the RE. The vector $\tilde{\epsilon}_t$ also depends on θ, so we have to assume that $E\tilde{\epsilon}_t = 0$.

However, the estimate of $\hat{\theta}_{t|t}$ will change over time as new information becomes available or as a consequence of policy reactions in the economy. So, as soon as we have implemented the control u_t, we will get a new realization of the state vector x_{t+1}, which

[2] See Kendrick (1981), chapter 6.

enables us to reestimate the parameter vector obtaining $\hat{\theta}_{t+1|t+1}$.

In the literature a number of procedures for such learning processes are described. For instance, ordinary least squares learning, filtering, or stochastic approximations. Here we will apply a Kalman filter to update the estimate $\hat{\theta}_{t|t}$ and the covariance matrix $\hat{\Sigma}^{\theta\theta}_{t|t}$. First it is necessary to project the covariance matrices to period $t+1$ using observation through period t, which produce the priors

$$\hat{\Sigma}^{xx}_{t+1|t} = f^{x}_{\theta t}\hat{\Sigma}^{\theta\theta}_{t|t}(f^{x}_{\theta t})' + \Sigma^{\epsilon\epsilon} \tag{17}$$

$$\hat{\Sigma}^{\theta x}_{t+1|t} = \hat{\Sigma}^{\theta\theta}_{t|t}(f^{x}_{\theta t}) \tag{18}$$

$$\hat{\Sigma}^{\theta\theta}_{t+1|t} = \hat{\Sigma}^{\theta\theta}_{t|t} \tag{19}$$

where

$$f^{x}_{\theta t} = \sum_{i=1}^{n} e_i x'_t a^i_\theta + \sum_{i=1}^{n} e_i u'_t b^i_\theta + \sum_{i=1}^{n} e_i c^i_\theta \tag{20}$$

Here the matrix $f^{x}_{\theta t}$ is the derivative of the system equations[3] with respect to the vector θ. In addition we also need an estimate of the state vector, which is

$$\hat{x}_{t+1|t} = A(\hat{\theta}_{t|t})x_t + B(\hat{\theta}_{t|t})u_t + c(\hat{\theta}_{t|t})_t + \sum_{j=1}^{k} D_j(\hat{\theta}_{t|t})\hat{x}_{t+j|t} \tag{21}$$

Next we update the parameter estimate and the covariance matrix for period $t+1$ using observation through period $t+1$, which produces the posterior

$$\hat{\theta}_{t+1|t+1} = \hat{\theta}_{t+1|t} + \hat{\Sigma}^{\theta x}_{t+1|t}(\hat{\Sigma}^{xx}_{t+1|t})^{-1}(x_{t+1} - \hat{x}_{t+1|t}) \tag{22}$$

$$\hat{\Sigma}^{\theta\theta}_{t+1|t+1} = \hat{\Sigma}^{\theta\theta}_{t+1|t} - \hat{\Sigma}^{\theta x}_{t+1|t}(\hat{\Sigma}^{xx}_{t+1|t})^{-1}\hat{\Sigma}^{x\theta}_{t+1|t} \tag{23}$$

so $\hat{\theta}_{t+1|t+1}$ is the new estimate of the parameter vector and $\hat{\Sigma}^{\theta\theta}_{t+1|t+1}$ the estimated covariance matrix. Starting with an initial estimates $\hat{\theta}_{0|0}$ and $\hat{\Sigma}^{\theta\theta}_{0|0}$ we can update the parameter vector each time new information on the state of the economy becomes available.

Step 0. Set $t=0$ and compute the estimate $\hat{\theta}_{t|t}$ and its corresponding covariance matrix $\hat{\Sigma}^{\theta\theta}_{t|t}$.

Step 1. Set the iteration counter $\nu = 0$.

Step 2. Set the instruments u^{ν}_i, $i = \{t, t+1, \ldots, T+s-1\}$

Step 3. Compute γ^{ν}_i, $i = \{t, t+1, \ldots, T+s-1\}$ and compute $\tilde{A}$, $\tilde{B}$ and $\forall\ i\ \tilde{c}_i$

Step 4. Apply standard LQ optimization method to compute a new set of optimal instruments $u^{\nu+1}_t$ using the equation below in place of equation (2a)

[3] For more detailed information please refer to Appendices L and M in Kendrick (1981).

$$\tilde{x}_{t+1}^{\nu+1} = \tilde{A}(\hat{\theta}_{t|t})\tilde{x}_t^{\nu+1} + \tilde{B}(\hat{\theta}_{t|t})u_t^{\nu+1} + \tilde{c}(\hat{\theta}_{t|t})_t^{\nu}$$

Step 5. Set $\nu = \nu + 1$ and go to Step 2 until convergence is reached on the RE part

Step 6. Estimate $\hat{\theta}_{t+1|t+1}$ and $\hat{\Sigma}^{\theta\theta}_{t+1|t+1}$ using equations (22)-(23)

Step 7. Set $t = t + 1$ and go to Step 1 if $t <= T$

Hence, Steps 1 to 5 outline the method for solving the stochastic policy framework for the RE part. Step 6 contains the learning part

6. An example

In this section we will present an example of the algorithm described in the previous section. Consider a simple macro model with output, x_t, consumption, c_t, investment, i_t, government expenditures, g_t, and taxes τ_t. The problem can then be stated as:

Find for the for the model

$$\begin{aligned} x_{t+1} &= c_{t+1} + i_{t+1} + g_{t+1} && (24)\\ c_{t+1} &= 0.8(x_t - \tau_t) + 200 && (25)\\ i_{t+1} &= 0.2E_t x_{t+2} - 0.1g_{t+1} + 100 + \epsilon_t && (26)\\ g_{t+1} &= u_t && (27)\\ \tau_{t+1} &= 0.25x_{t+1} && (28) \end{aligned}$$

with $x_0 = 1500$, a set of admissible control $U = \{u_0, u_1, \ldots, u_9\}$ to minimize the welfare loss function

$$J_T = \frac{1}{2}(x_{12} - 1600)^2 + \frac{1}{2}\sum_{t=0}^{11}\{(x_t - 1600)^2 + g_t^2\} \tag{29}$$

If we reduce the above model to one equation for output we get

$$x_{t+1} = 0.6x_t + 0.9u_t + 0.2E_t x_{t+2} + 300 + \epsilon_t \tag{30}$$

which leads to the augmented system

$$\begin{bmatrix} 1 & -0.2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_{t+1} \\ E_t x_{t+2} \end{bmatrix} = \begin{bmatrix} 0.6 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_t \\ x_{t+1} \end{bmatrix} + \begin{bmatrix} 0.9 \\ 0 \end{bmatrix} u_t + \begin{bmatrix} 300 \\ 0 \end{bmatrix} + \begin{bmatrix} \epsilon_t \\ 0 \end{bmatrix} \tag{31}$$

We will set $\Sigma^{\epsilon\epsilon} = 1$. Lets assume that the parameter $\theta = 0.9$ is unknown to the policy maker. However, he has the wrong initial estimate

$$\hat{\theta}_{0|0} = [0.8] \tag{32}$$

Furthermore, lets take as an estimate of the variance

$$\hat{\Sigma}^{\theta\theta}_{0|0} = \begin{bmatrix} 0.5 \end{bmatrix} \tag{33}$$

which is also arbitrarily chosen. By apply the QZ factorization we can compute the QZ decomposition

$$\Lambda = \begin{bmatrix} 1.0822 & -0.9136 \\ 0 & 0.1848 \end{bmatrix} \quad \Omega = \begin{bmatrix} 0.7546 & 0.3979 \\ 0 & 0.7952 \end{bmatrix} \tag{34}$$

$$Z = \begin{bmatrix} 0.8203 & -0.5719 \\ 0.5719 & 0.8203 \end{bmatrix} \quad Q = \begin{bmatrix} 0.6523 & 0.7580 \\ -0.7580 & 0.6523 \end{bmatrix} \tag{35}$$

so the eigenvalues are $\{0.6972, 4.3028\}$ and the ordering of the system is such that the unstable root 9.4721 is in the lower right corner. Due to the fact that θ only appears in the B matrix we get the following

$$\tilde{A} = \begin{bmatrix} 0.2966 & 0.5745 \\ 0.2068 & 0.4006 \end{bmatrix} \qquad \tilde{B}(\hat{\theta}_{0|0}) = \begin{bmatrix} 0.3955 \\ 0.2758 \end{bmatrix} \tag{36}$$

and for the initial period

$$\tilde{c}_0 = \begin{bmatrix} 195.66 \\ 615.09 \end{bmatrix} \tag{37}$$

$$\tilde{W} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \qquad R = \begin{bmatrix} 1 \end{bmatrix} \qquad \tilde{F} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{38}$$

$$\tilde{x}_0 = \begin{bmatrix} 1500 \\ 1500 \end{bmatrix} \qquad \tilde{\tilde{x}}_0 = \begin{bmatrix} 1600 \\ 0 \end{bmatrix} \tag{39}$$

Note, that we set E_0x_1 equal to 1500 for the first iteration. In order to deal with the boundary conditions of the RE part of the model we need the steady state of the system. Unfortunately, the policy maker cannot compute the steady state as the steady state depends on the unknown parameter vector θ. However, based on his initial estimate $\hat{\theta}_{0|0}$ he can make an estimate of the steady state. For $\hat{\theta}_{0|0}$ the steady state f the control vector is $u_\infty(\hat{\theta}_{0|0}) = 20.40$. Hence, $U^0 = \{20.40\ldots, 20.40\}$ is a good starting point for the instruments[4].

For computing the stochastic optimal instruments using equations (11)-(15), we need the expectations of [5]

$$E\{\tilde{B}'K_{t+1}\tilde{B}\} = \hat{\tilde{B}}'K_{t+1}\hat{\tilde{B}} + tr(K_{t+1}\Sigma^{\tilde{B}\tilde{B}}_{0|0}) \tag{40}$$

$\tilde{B}$ being a vector in this example. Given the fact that

$$\hat{\tilde{B}} = \begin{bmatrix} Z_{11} \\ Z_{12} \end{bmatrix} \Lambda_{11}^{-1} \hat{B} \tag{41}$$

the estimated covariance of $\tilde{B}$ at the initial period will be

$$\hat{\Sigma}_{0|0}^{\tilde{B}\tilde{B}} = \begin{bmatrix} Z_{11} \\ Z_{12} \end{bmatrix} \Lambda_{11}^{-1} \hat{\Sigma}_{0|0}^{BB} (\Lambda_{11}')^{-1} \begin{bmatrix} Z_{11} \\ Z_{12} \end{bmatrix}' \tag{42}$$

which is

$$\hat{\Sigma}_{0|0}^{\tilde{B}\tilde{B}} = \begin{bmatrix} Z_{11} \\ Z_{12} \end{bmatrix} \Lambda_{11}^{-1} \hat{\Sigma}_{0|0}^{\theta\theta} (\Lambda_{11}^{'})^{-1} \begin{bmatrix} Z_{11} \\ Z_{12} \end{bmatrix}' = \begin{bmatrix} 0.1222 & 0.0852 \\ 0.0852 & 0.0594 \end{bmatrix} \tag{43}$$

Now we have this covariance matrix we can compute the following solution

Table 1. Solution of the LQ optimization model with RE.

t	0	1	2	3	4	5	6	7	8	9	10	11	12
x_t	1500	1548	1571	1580	1584	1586	1585	1587	1587	1587	1583	1580	1573
u_t	37.80	28.89	23.12	21.04	19.95	19.45	19.76	19.08	18.87	18.47	16.95	11.57	
u_∞	20.40	18.86	19.03	19.18	19.20	19.19	19.32	19.23	19.20	19.22	19.30	19.30	

The results for the estimate $\hat{\theta}_{t|t}$ and θ are presented in the Figure 1. It is striking how quickly the algorithm is capable of adjusting to the regime switch.

7. Summary

In this paper we have presented a single agent stochastic optimization model that allows for rational expectations. Based on Sims's paper we have used a generalized eigenvalue method for solving the variables that involve the unstable roots. By using an iterative scheme, the reduced model can be fitted into a standard Linear-Quadratic framework that allows us to derive the stochastic optimal policy instruments for the model with rational expectations and learning.

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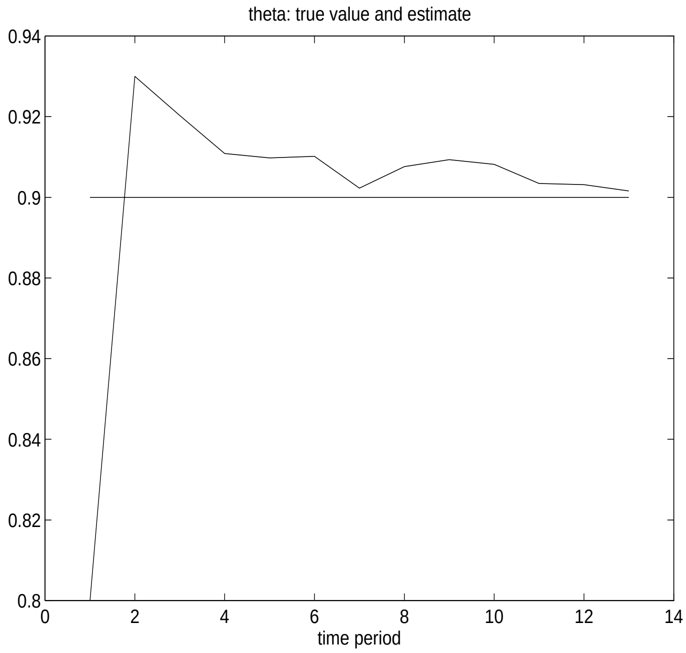
Figure 1



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Appendix A
The Relationship Between the Kalman Filter and the Ljung and Söderström Estimator

In a previous paper, Amman and Kendrick (1998c), we made use of the Ljung and Söderström (1983) parameter updating equation while in this paper we use the Kalman filter for the same purpose. This appendix uses the developments in Kendrick (1981) from the Kalman filter side and in Ljung and Söderström (1983) from that side to begin to elucidate the relationship between the two estimation methods.

In the following all equation references of the form "equation (10-xx)" will be to Chapter 10 in Kendrick (1981). Begin with the parameter updating equation (10-68).

$$\hat{\theta}_{t+1|t+1} = \hat{\theta}_{t+1|t} + \Sigma^{\theta x}_{t+1|t} H'_{t+1} S^{-1}_{t+1} [y_{t+1} - H_{t+1} \hat{x}_{t+1|t}] \tag{A-1}$$

where

$\hat{\theta}_{t+1\|t+1}$	=	the stacked-up parameter vector as estimated at period $t+1$ with data through period $t+1$
$\hat{\theta}_{t+1\|t}$	=	the stacked-up parameter vector as estimated at period $t+1$ with data through period t
$\Sigma^{\theta x}_{t+1\|t}$	=	covariance matrix between the state vector and the parameter vector
H_{t+1}	=	state variable matrix in the measurement relationship
S_{t+1}	=	an auxiliary matrix to be defined below
y_{t+1}	=	measurement vector
$\hat{x}_{t+1\|t}$	=	projected state vector at period $t+1$ with data through period t

and from equation (10-69)

$$S_{t+1} = H_{t+1} \Sigma^{xx}_{t+1|t} H'_{t+1} + \tilde{R}_{t+1} \tag{A-2}$$

where

$\Sigma^{xx}_{t+1\|t}$	=	projected covariance matrix for the state vector at period t+1 with data through period t
$\tilde{R}_{t+1}$	=	covariance of the additive noise in the measurement relationship

For the problem at hand all states are measured without error so

$$H_t = I \tag{A-3}$$

and

$$\tilde{R}_t = 0 \tag{A-4}$$

Thus the use of equations (A-3) and (A-4) in equation (A-2) yields

$$S_{t+1} = \Sigma^{xx}_{t+1|t} \tag{A-5}$$

Next we use the parameter evolution equation (10-9), i.e.

$$\theta_{t+1} = D_t \theta_t + \eta_t \tag{A-6}$$

where

$$\begin{aligned} D &= \text{the parameter evolution matrix} \\ \eta_t &= \text{additive noise in the parameter evolution equation} \\ &\quad \text{with } \eta_t \sim i.i.d.\ N(0, G) \end{aligned} \tag{A-7}$$

For the case at hand assume

$$\begin{aligned} D &= I \\ G &= 0 \end{aligned}$$

That is assume that the true values of the parameters are not time varying. Of course the estimates of these parameters will be time varying but the true values will remain constant.

Next consider the covariance projection equation (10-56) and use the assumption that $D = I$. This yields

$$\Sigma^{\theta x}_{t+1|t} = \Sigma^{\theta x}_{t|t} A'_t + \Sigma^{\theta\theta}_{t|t} (f^x_{\theta t})' \tag{A-8}$$

Then consider the updating equation for the covariance of the state variables and the parameter vector, i.e. equation (10-60) with the assumption used above from the measurement equation, i.e. $H = I$ to obtain

$$\Sigma^{\theta x}_{t+1|t+1} = \Sigma^{\theta x}_{t+1|t} [I - S^{-1}_{t+1} \Sigma^{xx}_{t+1|t}] \tag{A-9}$$

Substitution of equation (A-5) into equation (A-8) yields

$$\begin{aligned} \Sigma^{\theta x}_{t+1|t+1} &= \Sigma^{\theta x}_{t+1|t} [I - (\Sigma^{xx}_{t+1|t})^{-1} \Sigma^{xx}_{t+1|t}] \\ &= 0 \end{aligned} \tag{A-10}$$

So at this stage we have

$$\Sigma^{\theta x}_{t+1|t+1} = 0 \text{ and therefore } \Sigma^{\theta x}_{t|t} = 0 \tag{A-11}$$

but

$$\Sigma^{\theta x}_{t+1|t} \neq 0 \tag{A-12}$$

Consider also that

$$\Sigma_{t|t}^{xx} = 0 \tag{A-13}$$

since there is no measurement error. Thus using this result and equation (A-10) in the state variable covariance projection equation (10-55) we obtain

$$\Sigma_{t+1|t}^{xx} = f_{\theta_t}^{x} \Sigma_{t|t}^{\theta\theta} (f_{\theta t}^{x})' + Q \tag{A-14}$$

Now return to equation (A-1) and use the assumption that $H = I$ to obtain

$$\hat{\theta}_{t+1|t+1} = \hat{\theta}_{t+1|t} + \Sigma_{t+1|t}^{\theta x} S_{t+1}^{-1} [y_{t+1} - \hat{x}_{t+1|t}] \tag{A-15}$$

Next substitute equation (A-5) into equation (A-14) to obtain

$$\hat{\theta}_{t+1|t+1} = \hat{\theta}_{t+1|t} + \Sigma_{t+1|t}^{\theta x} (\Sigma_{t+1|t}^{xx})^{-1} [y_{t+1} - \hat{x}_{t+1|t}] \tag{A-16}$$

Now we can substitute equations (A-7) and (A-13) into equation (A-15) to obtain

$$\hat{\theta}_{t+1|t+1} = \hat{\theta}_{t+1|t} + \{\Sigma_{t|t}^{\theta x} A_t' + \Sigma_{t|t}^{\theta\theta} (f_{\theta t}^{x})'\} \{f_{\theta t}^{x} \Sigma_{t|t}^{\theta\theta} (f_{\theta t}^{x})' + Q\}^{-1} [y_{t+1} - \hat{x}_{t+1|t}] \tag{A-17}$$

From equation (A-10) $\Sigma_{t|t}^{\theta x} = 0$ so equation (A-16) becomes

$$\hat{\theta}_{t+1|t+1} = \hat{\theta}_{t+1|t} + \{\Sigma_{t|t}^{\theta\theta} (f_{\theta t}^{x})'\} \{f_{\theta t}^{x} \Sigma_{t|t}^{\theta\theta} (f_{\theta t}^{x})' + Q\}^{-1} [y_{t+1} - \hat{x}_{t+1|t}] \tag{A-18}$$

Next consider the y_{t+1} term in equation (A-17). This is the measurement vector which is determined by equation (10-8) i.e.

$$y_{t+1} = H_{t+1} x_{t+1} + w_{t+1} \tag{A-19}$$

and for the case at hand of no measurement error we have $H = I$ and $w = 0$ so equation (A-18) becomes

$$y_{t+1} = x_{t+1} \tag{A-20}$$

Substitution of equation (A-19) into equation (A-17) then yields

$$\hat{\theta}_{t+1|t+1} = \hat{\theta}_{t+1|t} + \{\Sigma_{t|t}^{\theta\theta} (f_{\theta t}^{x})'\} \{f_{\theta t}^{x} \Sigma_{t|t}^{\theta\theta} (f_{\theta t}^{x})' + Q\}^{-1} [x_{t+1} - \hat{x}_{t+1|t}] \tag{A-21}$$

Next consider the $\hat{x}_{t+1|t}$ term in the equation above by using the state vector projection equation (10-52) along with the result from equation (A-10) above that $\Sigma_{t|t}^{\theta x} = 0$. This yields

$$\hat{x}_{t+1|t} = A(\hat{\theta}_{t|t}) \hat{x}_{t|t} + B(\hat{\theta}_{t|t}) \hat{u}_{t|t}^{\tau} + c(\hat{\theta}_{t|t}) \tag{A-22}$$

Also from Appendix L in Kendrick (1981) we take the notation

$$f^{x} = A(\hat{\theta}_{t|t}) \hat{x}_{t|t} + B(\hat{\theta}_{t|t}) \hat{u}_{t|t}^{\tau} + c(\hat{\theta}_{t|t}) \tag{A-23}$$

Thus from equations (A-21) and (A-22) we have

$$\Sigma^{\theta\theta}_{t+1|t+1} = \Sigma^{\theta\theta}_{t+1|t} - \Sigma^{\theta\theta}_{t|t}(f^x_{\theta t})'\{f^x_{\theta t}\Sigma^{\theta\theta}_{t|t}(f^x_{\theta t})' + Q\}^{-1} f^x_{\theta t}(\Sigma^{\theta\theta}_{t|t})' \tag{A-33}$$

Also from equation (10-57) with $D = I$ and $G = 0$ we have

$$\Sigma^{\theta\theta}_{t+1|t} = \Sigma^{\theta\theta}_{t|t} \tag{A-34}$$

Substitution of equation (A-33) into equation (A-32) yields

$$\Sigma^{\theta\theta}_{t+1|t+1} = \Sigma^{\theta\theta}_{t|t} - \Sigma^{\theta\theta}_{t|t}(f^x_{\theta t})'\{f^x_{\theta t}\Sigma^{\theta\theta}_{t|t}(f^x_{\theta t})' + Q\}^{-1} f^x_{\theta t}(\Sigma^{\theta\theta}_{t|t})' \tag{A-35}$$

Now we are in a positon to compare the Kalman filter approach and the Ljung and Söderström approach to the parameter covariance updating equation. The Ljung and Soderstrom equation from Amman and Kendrick (1998c) equation (14) is

$$\hat{\Sigma}_{t+1} = \hat{\Sigma}_t + \alpha_t(H_t' H_t - \hat{\Sigma}_t) \tag{A-36}$$

and note that the H matrix in the Ljung and Söderström notation is not the same as in the Kendrick (1981) notation but is rather more like f^x_θ in the Kendrick notation. Thus a comparison of equations (A-34) and (A-35) shows that the Kalman filter approach and the Ljung and Söderström approaches are substantially different in the parameter covariance updating equations.

In summary, if the derivations above hold up to further scrutiny, one can say that the Kalman filter and Ljung and Söderström stochastic approximation approaches are similar in the parameter updating equations but dissimilar in the parameter covariance updating equations.

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